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GLOBAL SOLUTIONS OF ABSTRACT QUASI-LINEAR EVOLUTION EQUATIONS OF "HYPERBOLIC" TYPE

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ABSTRACT

The problem of existence and uniqueness of global classical solutions of abstract quasi-linear evolution equations is considered in a general Banach space. The results obtained here are applied to the initial value problems for hyperbolic partial differential equations.

This paper is concerned with the abstract quasi-linear evolution equation

$$\text{(QE)} \qquad \begin{cases} u'(t) = A(u(t))u(t) & \text{for } t \geq 0, \\ u(0) = u_0 \end{cases}$$

in a real Banach space Z, where $\{A(w) \colon w \in Y\}$ is a family of closed linear operators in Z and Y is another real Banach space which is densely and continuously imbedded in Z.

There are at least two different operator-theoretical approaches to the existence problem for quasi-linear hyperbolic partial differential equations. One is the theory of quasi-contractive nonlinear semigroups, which was applied to first order quasi-linear equations in several space variables by Crandall [1]. However, this method breaks down for a broad class of systems, since the quasi-contractive continuity cannot be expected in that case. A new fully nonlinear existence theory covering the quasi-linear examples has been presented by Crandall and Souganidis [2]. An attempt to develop the theory of nonlinear semigroups of Lipschitz continuous operators and not quasi-contractions is found in Kobayashi and Tanaka [8]. The other is the theory of abstract quasi-linear evolution equations initiated by Kato [5], which has been constantly recognized to be important from

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both theoretical and practical points of view. Most of the literature dealing with such quasi-linear evolution equations is devoted to the study of local existence of classical solutions. Among others, Kobayasi and Sanekata [9] succeeded in proving an existence theorem of local classical solutions without assuming the reflexivity of Z and Y, and their result was improved by Kato [6] so that it can be applied to the system of first order quasi-linear equation in $C(\mathbb{R}^m)$. So far sufficient conditions have been investigated extensively for quasi-linear evolution equations to possess local classical solutions. However, it seems to us that very little is known about sufficient conditions on $\{A(w)\colon w \in Y\}$ for the classical solutions to exist globally in time, while there are several works concerning the global existence of solutions of quasi-linear hyperbolic partial differential equations such as the wave equation of Kirchhoff type.

We are here interested in developing the latter abstract theory so that it is applicable to the problem of existence and uniqueness of global solutions of quasi-linear hyperbolic systems; hence our purpose is to discuss the problem of global existence of classical solutions of quasi-linear evolution equations of the type (QE). Equation (QE) may have only local classical solutions provided that $A(w)$ is local quasi-dissipative for each $w \in Y$, and it is necessary to consider the growth of classical solutions. Here we employ a nonnegative continuous functional φ on Y to define the local quasi-dissipativity of $A(w)$ and specify the growth of a classical solution u of (QE) in terms of the real-valued function $\varphi(u(\cdot))$. In case of concrete partial differential equations the use of such a functional φ corresponds to a priori estimates or energy estimates which ensure the global existence of the solutions as well as their asymptotic properties. It should be noted that the idea of the localization with respect to φ is affected by the Lyapunov method and that the present paper is similar in spirit to Oharu and Takahashi [11] discussing nonlinear semigroups associated with semilinear evolution equations. The Lyapunov method for nonlinear semigroups is found in Pazy [14] and Walker [15].

In Section 1 we formulate typical hypotheses on $A(w)$ in a local sense by using a functional φ and investigate the uniqueness of classical solutions of (QE). This section contains the statement of main theorem and some of basic properties of maximal solutions of scalar ordinary differential equations used in later arguments. Section 2 provides the construction of approximate solutions for (QE) where the "semi-implict" discrete scheme

$$\begin{cases} (u_i - u_{i-1})/(t_i - t_{i-1}) = A(u_{i-1})u_i \quad \text{for } i = 1, 2, \ldots, \\ 0 = t_0 < t_1 < t_2 < \cdots < t_i < \cdots \end{cases}$$

is used instead of the "fully implicit" discrete scheme.

Section 3 discusses the convergence of approximate solutions constructed in Section 2. The problem of this kind has been studied by Crandall and Souganidis [2]. Our result (Theorem 3.1) is different from theirs in that it shows the convergence of solutions of the discrete problem in a "good" subspace Y of Z. Typical examples such as damped extensible beam equations and quasi-linear wave equations are presented in final Section 4 to illustrate our abstract theory.

1. Preliminaries and main result

In this section we state the main result of this paper. We start with three real Banach spaces $Y \subset X \subset Z$, with all the inclusions continuous and dense; hence there exist $c_X > 0$ and $c_Y > 0$ such that $\|x\|_Z \le c_X\|x\|_X$ for $x \in X$, and $\|y\|_X \le c_Y\|y\|_Y$ for $y \in Y$. It is assumed that Z and X have the same topology on a bounded set of Y in the following sense: Given any bounded subset B of Y and $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in B$ with $\|x - y\|_Z < \delta$ implies $\|x - y\|_X < \varepsilon$. We consider a continuous functional $\varphi\colon Y \to [0, \infty)$ such that

(φ1) for each $\alpha \ge 0$ the set $Y_\alpha = \{w \in Y\colon \varphi(w) \le \alpha\}$ is bounded in Y,

(φ2) φ is bounded on each bounded subset of Y.

We now set up basic hypotheses in a local sense on the operators $A(w)$ appearing in (QE) by means of the functional φ.

(N) For each $w \in Y$ there exists a norm $\|\cdot\|_{(w)}$ in Z with the following properties:

(N1) For each $\alpha \ge 0$ there exists $M_Z(\alpha) \ge 1$ such that

$$M_Z(\alpha)^{-1}\|z\|_Z \le \|z\|_{(w)} \le M_Z(\alpha)\|z\|_Z \tag{1.1}$$

for $z \in Z$ and $w \in Y_\alpha$.

(N2) For each $\alpha \ge 0$ there exists $L_Z(\alpha) \ge 0$ such that

$$\|z\|_{(w)} \le \|z\|_{(\hat{w})}(1 + L_Z(\alpha)\|w - \hat{w}\|_X) \tag{1.2}$$

for $z \in Z$ and $w, \hat{w} \in Y_\alpha$.

It should be noted here that Hughes *et al.* [4] first proposed the equivalent norms satisfying conditions (N1) and (N2), and established the abstract theory which is applied to second-order quasi-linear hyperbolic systems on $\mathbb{R}^m$.

(A) The family $\{A(w)\colon w \in Y\}$ of closed linear operators in Z satisfies the following conditions:

(A1) For each $\alpha \ge 0$ there exists $\omega(\alpha) \ge 0$ such that

$$A(w) \in G(Z_{(w)}, 1, \omega(\alpha))$$

for $w \in Y_\alpha$, where $Z_{(w)}$ denotes the Banach space Z with the norm $\|\cdot\|_{(w)}$. Here and subsequently, $\mathfrak{A} \in G(\mathfrak{X}, M, \beta)$ is written for the infinitesimal generator $\mathfrak{A}$ of a semigroup $\{T(t)\colon t \geq 0\}$ of class (C_0) on $\mathfrak{X}$ satisfying $\|T(t)\|_{\mathfrak{X},\mathfrak{X}} \leq Me^{\beta t}$ for $t \geq 0$.

(A2) There exist an isomorphism S of Y onto Z and a family $\{B(w)\colon w \in Y\}$ in $B(Z)$ such that

$$SA(w)S^{-1} = A(w) + B(w)$$

for $w \in Y$, where the family $\{B(w)\colon w \in Y\}$ satisfies the following properties: For each $\alpha \geq 0$ there exists $L_B(\alpha) \geq 0$ such that

$$\|B(w) - B(\hat{w})\|_{Z,Z} \leq L_B(\alpha)\|w - \hat{w}\|_Y \tag{1.3}$$

for $w, \hat{w} \in Y_\alpha$.

(A3) For $w \in Y$, $D(A(w)) \supset Y$, and $A \in C(Y; B(Y, X))$. For each $\alpha \geq 0$ there exist $M_A(\alpha) \geq 0$ and $L_A(\alpha) \geq 0$ such that

$$\|A(w)\|_{Y,X} \leq M_A(\alpha) \qquad \text{for } w \in Y_\alpha, \tag{1.4}$$

$$\|A(w) - A(\hat{w})\|_{Y,Z} \leq L_A(\alpha)\|w - \hat{w}\|_Z \qquad \text{for } w, \hat{w} \in Y_\alpha. \tag{1.5}$$

Remark 1.1: (i) From (1.3) and property (φ1) it follows readily that for each $\alpha \geq 0$, there exists $M_B(\alpha) \geq 0$ such that

$$\|B(w)\|_{Z,Z} \leq M_B(\alpha) \qquad \text{for } w \in Y_\alpha. \tag{1.6}$$

(ii) For each $w \in Y_\alpha$, we have

$$A(w) + B(w) \in G(Z_{(w)}, 1, \overline{\omega}(\alpha)), \tag{1.7}$$

$$(I - hA(w))^{-1}w = S^{-1}(I - h(A(w) + B(w)))^{-1}Sw \tag{1.8}$$

for $h > 0$ with $h\overline{\omega}(\alpha) < 1$, where we set $\overline{\omega}(\alpha) = \omega(\alpha) + M_Z(\alpha)^2 M_B(\alpha)$.

The first assertion is proved by the perturbation theorem and the following estimate which follows from (1.1) and (1.6):

$$\|B(w)\|_{Z_{(w)},Z_{(w)}} \leq M_Z(\alpha)^2 M_B(\alpha).$$

The second assertion follows from condition (A2).

(iii) For each $w \in Y$, there exists $h_0 > 0$ such that $(I - hA(w))^{-1}w \in Y$ for $h \in (0, h_0]$, and $\lim_{h\downarrow 0}(I - hA(w))^{-1}w = w$ in Y. This fact follows from (ii).

Because of the localized conditions stated above, problem (QE) may have only local classical solutions by the theory established in [6] and [9]. Hereafter we mean by a **classical solution** u to (QE) on $J = [0, \tau]$ or $[0, \tau)$ with $0 < \tau \le \infty$ that $u \in C(J\colon Y) \cap C^1(J\colon X)$ and the (QE) is satisfied for $t \in J$. A classical solution to (QE) on $[0, \infty)$ is called a **global classical solution** to (QE).

Throughout this paper we may assume $c_X = c_Y = 1$ without loss of generality. We recall the following uniqueness theorem of classical solutions to (QE) with proof.

THEOREM 1.1: *For each $T > 0$, the* (QE) *has at most one classical solution on* $[0, T]$.

Proof: Let $T > 0$ be fixed arbitrarily. By u and v we denote two classical solutions to (QE) on $[0, T]$, and set $r_0 = \sup\{\|u(t)\|_Y \vee \|v(t)\|_Y : t \in [0, T]\}$. Condition ($\varphi$2) implies $\alpha_0 = \sup\{\varphi(w)\colon \|w\|_Y \le r_0\} < \infty$. To prove $u = v$ on $[0, T]$, consider the function $\phi(t) = \|u(t) - v(t)\|_{(u(t))}$ on $[0, T]$. We first show that ϕ is continuous on $[0, T]$. To this end, let $s, t \in [0, T]$. By (1.2) we have

$$\begin{aligned}\phi(t) - \phi(s) \le (&\|u(t) - v(t)\|_{(u(s))} - \|u(s) - v(s)\|_{(u(s))}) \\ &+ \|u(t) - v(t)\|_{(u(s))} L_Z(\alpha_0) \|u(t) - u(s)\|_X.\end{aligned}$$

By (1.1) the right-hand side is bounded by

$$M_Z(\alpha_0)\|u(t) - v(t) - (u(s) - v(s))\|_Z + 2r_0 M_Z(\alpha_0) L_Z(\alpha_0) \|u(t) - u(s)\|_X.$$

This implies the continuity of ϕ on $[0, T]$. Now, we compute $D_-\phi(t)$, where

$$D_-\phi(t) = \liminf_{h \downarrow 0} (\phi(t) - \phi(t - h))/h$$

for $t \in (0, T]$. Let $t \in (0, T]$ and $h > 0$ such that $t - h \in [0, T]$. Then $(\phi(t) - \phi(t - h))/h$ is written as

$$\begin{aligned}(1.9)\quad &(\|u(t) - v(t)\|_{(u(t))} - \|u(t-h) - v(t-h)\|_{(u(t))})/h \\ &+ (\|u(t-h) - v(t-h)\|_{(u(t))} - \|u(t-h) - v(t-h)\|_{(u(t-h))})/h,\end{aligned}$$

and the first term on the right-hand side tends to $[u(t) - v(t), u'(t) - v'(t)]_{(u(t))}$ as $h \downarrow 0$, where $[x, y]_{(u(t))}$ is defined by

$$[x, y]_{(u(t))} = \lim_{h \downarrow 0} (\|x\|_{(u(t))} - \|x - hy\|_{(u(t))})/h.$$

By condition (A1) and (1.5) we have

$$\begin{aligned}
&[u(t)-v(t), u'(t)-v'(t)]_{(u(t))} \\
&= [u(t)-v(t), A(u(t))(u(t)-v(t)) + (A(u(t))-A(v(t)))v(t)]_{(u(t))} \\
&\le \omega(\alpha_0)\|u(t)-v(t)\|_{(u(t))} + M_Z(\alpha_0)^2 L_A(\alpha_0)\|u(t)-v(t)\|_{(u(t))} r_0.
\end{aligned}$$

By using (1.2), the last term on the right-hand side of (1.9) is majorized by

$$\|u(t-h)-v(t-h)\|_{(u(t-h))} L_Z(\alpha_0)\|u(t)-u(t-h)\|_X/h,$$

which tends to $\|u(t)-v(t)\|_{(u(t))} L_Z(\alpha_0)\|u'(t)\|_X$ as $h \downarrow 0$. We have $\|u'(t)\|_X \le \|A(u(t))\|_{Y,X}\|u(t)\|_Y \le M_A(\alpha_0)r_0$. It follows that $D_-\phi(t) \le \beta_0\phi(t)$ for $t \in (0,T]$, where $\beta_0 = \omega(\alpha_0) + M_Z(\alpha_0)^2 L_A(\alpha_0)r_0 + L_Z(\alpha_0)M_A(\alpha_0)r_0$. Solving this differential inequality we find $\phi(t) \le \exp(\beta_0 t)\phi(0)$ for $t \in [0,T]$. Since $\phi(0) = 0$ we have $u = v$ on $[0,T]$. ∎

In our setting, problem (QE) may have only local classical solutions, and it is necessary to consider the growth of classical solutions. Here we specify the growth of a classical solution $u(\cdot)$ of (QE) by means of the function $\varphi(u(\cdot))$. A nonnegative continuous function g on $[0,\infty)$ is called a **comparison function** if there is an $\alpha_0 > 0$ such that $\tau(\alpha_0) = \infty$, where $[0, \tau(\alpha))$ denotes the interval of existence of the non-extensible maximal solution $m(t;\alpha)$ of the initial value problem

$$(1.10) \qquad r'(t) = g(r(t)) \quad \text{for } t \ge 0, \qquad \text{and} \qquad r(0) = \alpha.$$

We choose such a comparison function g and consider global classical solution $u(\cdot)$ of (QE) satisfying the growth condition

$$(1.11) \qquad \varphi(u(t)) \le m(t;\varphi(u_0)) \quad \text{for } t \ge 0$$

for the initial data $u_0 \in Y_{\alpha_0}$. We give here two typical examples of comparison functions.

Example 1.1: (i) Let $a, b \ge 0$. A function g defined by $g(r) = ar + b$ for $r \ge 0$ is a comparison function, and the associated non-extensible maximal solution of (1.10) is given by $m(t;\alpha) = e^{at}\alpha + b\int_0^t e^{a(t-s)}\,ds$ for $t \ge 0$. Note that $\tau(\alpha) = \infty$ for all $\alpha \ge 0$.

(ii) Another example of comparison function is given by a function g of the form

$$(1.12) \qquad g(r) = ((\rho(r) - c)r) \vee 0$$

where ρ be a nonnegative continuous function with $\rho(0) = 0$, and $c > 0$. Indeed, if we choose $\alpha_0 > 0$ so that $\rho(r) \leq c$ for $r \in [0, \alpha_0]$ then for each $\alpha \in [0, \alpha_0)$, we have $\tau(\alpha) = \infty$ and $m(t; \alpha) = \alpha$ for $t \geq 0$.

Throughout this paper we assume that conditions (N) and (A) are satisfied. The main result of this paper is given by

MAIN THEOREM: *Suppose that the following condition* (G) *is satisfied.*

(G) *There is a comparison function g with $\tau(\alpha_0) = \infty$ such that*

$$\liminf_{h \downarrow 0} (\varphi((I - hA(w))^{-1}w) - \varphi(w))/h \leq g(\varphi(w)) \quad \text{for } w \in Y.$$

Then for each $u_0 \in Y_{\alpha_0}$, there is a unique global classical solution u to (QE) *satisfying the growth condition* (1.11).

We conclude this section by listing up some basic properties of maximal solutions used later.

For each $\varepsilon > 0$ we write $m_\varepsilon(t; \alpha)$ for the non-extensible maximal solution of the initial value problem

$$r'(t) = g_\varepsilon(r(t)) \quad \text{for } t \geq 0, \quad \text{and} \quad r(0) = \alpha,$$

where g_ε is defined by $g_\varepsilon(r) = g(r) + \varepsilon$ for $r \geq 0$. The maximal interval of existence of $m_\varepsilon(t; \alpha)$ is denoted by $[0, \tau_\varepsilon(\alpha))$.

PROPOSITION 1.2: *The following assertions hold:*

(i) *If $\alpha \geq \alpha_0$ and $\varepsilon \geq \varepsilon_0$ then $\tau_\varepsilon(\alpha) \leq \tau_{\varepsilon_0}(\alpha_0)$ and $m_\varepsilon(t; \alpha) \geq m_{\varepsilon_0}(t; \alpha_0)$ for $t \in [0, \tau_\varepsilon(\alpha))$.*

(ii) *As $\varepsilon \downarrow \varepsilon_0$ and $\alpha \downarrow \alpha_0$, we have $\tau_\varepsilon(\alpha) \uparrow \tau_{\varepsilon_0}(\alpha_0)$ and $m_\varepsilon(t; \alpha) \downarrow m_{\varepsilon_0}(t; \alpha_0)$ uniformly on every compact subinterval of $[0, \tau_{\varepsilon_0}(\alpha_0))$.*

(iii) *If $s \in [0, \tau_\varepsilon(\alpha))$ then $\tau_\varepsilon(m_\varepsilon(s; \alpha)) = \tau_\varepsilon(\alpha) - s$ and*

$$m_\varepsilon(t + s; \alpha) = m_\varepsilon(t; m_\varepsilon(s; \alpha)) \quad \text{for } t \in [0, \tau_\varepsilon(\alpha) - s).$$

(iv) *If $\varepsilon > \varepsilon_0$ then for each $\alpha \geq 0$, $m_\varepsilon(t; \alpha) > m_{\varepsilon_0}(t; \alpha)$ for $t \in (0, \tau_\varepsilon(\alpha))$.*

Proof: The elementary facts (i) through (iii) have been already proved in [8]. To prove (iv), let $\alpha \geq 0$ and $\varepsilon > \varepsilon_0$. A continuously differentiable function f on $[0, \tau_\varepsilon(\alpha))$ defined by

$$f(t) = m_\varepsilon(t; \alpha) - m_{\varepsilon_0}(t; \alpha)$$

satisfies $f'(0) = g(\alpha) + \varepsilon - (g(\alpha) + \varepsilon_0) > 0$. By the continuity of f' there is a $t_0 \in (0, \tau_\varepsilon(\alpha))$ such that $f'(\xi) > 0$ for $\xi \in [0, t_0]$. Since $f(0) = 0$ we have by the mean value theorem, $f(t) > 0$ for $t \in (0, t_0]$; namely $m_\varepsilon(t; \alpha) > m_{\varepsilon_0}(t; \alpha)$ for $t \in (0, t_0]$. For $t \in [t_0, \tau_\varepsilon(\alpha))$, we find by an easy computation

$$m'_\varepsilon(t;\alpha) > g_{\varepsilon_0}(m_\varepsilon(t;\alpha)) \quad \text{and} \quad m'_{\varepsilon_0}(t;\alpha) = g_{\varepsilon_0}(m_{\varepsilon_0}(t;\alpha)).$$

Since $m_\varepsilon(t_0; \alpha) > m_{\varepsilon_0}(t_0; \alpha)$, we have $m_\varepsilon(t; \alpha) > m_{\varepsilon_0}(t; \alpha)$ for $t \in [t_0, \tau_\varepsilon(\alpha))$, by [10, Theorem 1.2.1]. ∎

2. Construction of "semi-implicit" discrete approximations

The main result in this section is given by the following theorem which ensures the existence of "semi-implicit" discrete approximations of (QE).

THEOREM 2.1: *Suppose that condition* (G) *holds. Let* $\varepsilon > 0$ *and* $u_0 \in Y$. *Then there exists a sequence* $\{(t_i, u_i)\}_{i=0}^\infty$ *in* $[0, \infty) \times Y$ *such that it satisfies the following conditions:*

(i) $0 = t_0 < t_1 < \cdots < t_i < \cdots < \tau_\varepsilon(\varphi(u_0))$;
(ii) $t_i - t_{i-1} \le \varepsilon$ *for* $i = 1, 2, \ldots$;
(iii) $(u_i - u_{i-1})/(t_i - t_{i-1}) = A(u_{i-1})u_i$ *for* $i = 1, 2, \ldots$;
(iv) $\varphi(u_i) \le m_\varepsilon(t_i; \varphi(u_0))$ *for* $i = 1, 2, \ldots$;
(v) $\lim_{i\to\infty} t_i = \tau_\varepsilon(\varphi(u_0))$.

We prove four lemmas needed for the proof of Theorem 2.1.

LEMMA 2.2: *Suppose that condition* (G) *holds. Then for each* $\varepsilon > 0$ *and* $w \in Y$, *there is a null sequence* $\{\lambda_n\}_{n=1}^\infty$ *of positive numbers such that*

$$\varphi((I - \lambda_n A(w))^{-1} w) \le m_\varepsilon(\lambda_n; \varphi(w))$$

for $n \ge 1$.

Proof: Let $\varepsilon > 0$ and $w \in Y$. By condition (G) there is a null sequence $\{\lambda_n\}_{n=1}^\infty$ of positive numbers such that $\varphi((I - \lambda_n A(w))^{-1} w) \le (g(\varphi(w)) + \varepsilon/2)\lambda_n + \varphi(w)$ for $n \ge 1$. Without loss of generality, it may be assumed that $\lambda_n \in [0, 1]$ for all $n \ge 1$. Now, let us define $r_n(t) = (g(\varphi(w)) + \varepsilon/2)t + \varphi(w)$ for $t \in [0, \lambda_n]$. We wish to prove

$$r_n(t) \le m_\varepsilon(t; \varphi(w)) \quad \text{for } t \in [0, \lambda_n] \cap [0, \tau_\varepsilon(\varphi(w))). \tag{2.1}$$

To this end, we differentiate $r_n(t)$ and use the estimate

$$|r_n(t) - \varphi(w)| \leq (g(\varphi(w)) + \varepsilon/2)\lambda_n$$

for $t \in [0, \lambda_n]$. This yields

$$r_n'(t) \leq g(r_n(t)) + \rho(g(\varphi(w)) + \varepsilon/2 + \varphi(w); (g(\varphi(w)) + \varepsilon/2)\lambda_n) + \varepsilon/2$$

for $t \in [0, \lambda_n]$, where $\rho(M; r) = \sup\{|g(t) - g(s)|: 0 \leq t, s \leq M, |t - s| \leq r\}$. Clearly, $\lim_{r\downarrow 0} \rho(M; r) = 0$ for each $M \geq 0$; hence

$$\rho(g(\varphi(w)) + \varepsilon/2 + \varphi(w); (g(\varphi(w)) + \varepsilon/2)\lambda_n) \leq \varepsilon/2$$

for sufficiently large n. It follows that

$$r_n'(t) \leq g_\varepsilon(r_n(t)) \quad \text{for } t \in [0, \lambda_n], \quad \text{and} \quad r_n(0) = \varphi(w),$$

which implies (2.1) by the comparison theorem. We have $\lambda_n < \tau_\varepsilon(\varphi(w))$ for sufficiently large n, since $\tau_\varepsilon(\varphi(w)) > 0$. The desired claim is then proved by substituting $t = \lambda_n$ into (2.1). ∎

LEMMA 2.3: *Suppose that $\{\widetilde{A}(w): w \in Y\}$ is a family of closed linear operators in Z satisfying the following condition:*

For each $\alpha \geq 0$, there exists $\tilde{\omega}(\alpha) \geq 0$ such that

$$\widetilde{A}(w) \in G(Z_{(w)}, 1, \tilde{\omega}(\alpha)) \qquad \text{for } w \in Y_\alpha. \tag{2.2}$$

Let $\alpha > 0$ and assume that a sequence $\{t_l\}_{l=0}^\infty$ of nonnegative numbers and a sequence $\{w_l\}_{l=0}^\infty$ in Y_α satisfy the following conditions:

(i) $0 = t_0 < t_1 < t_2 < \cdots$;

(ii) *there exists $L \geq 0$ such that*

$$\|w_l - w_{l-1}\|_X \leq L(t_l - t_{l-1}) \qquad \text{for } l = 1, 2, \ldots.$$

If there exists $k_0 \geq 0$ such that $(t_l - t_{l-1})\tilde{\omega}(\alpha) \leq 1/2$ for $l \geq k_0 + 1$, then

$$\left\| \prod_{l=k+1}^{i} (I - (t_l - t_{l-1})\widetilde{A}(w_{l-1}))^{-1} \right\|_{Z,Z} \leq M(\alpha)\exp(\tilde{\beta}(\alpha)(t_i - t_k))$$

for $i \geq k$ and $k \geq k_0$, where $M(\alpha) = M_Z(\alpha)^2$ and $\tilde{\beta}(\alpha) = 2\tilde{\omega}(\alpha) + L_Z(\alpha)L$.

Proof: Let $z \in Z$ and $k \geq k_0$, and then set

$$a_i = \left\| \prod_{l=k+1}^{i} (I - h_l \widetilde{A}(w_{l-1}))^{-1} z \right\|_{(w_i)}$$

for $i \geq k$, where $h_l = t_l - t_{l-1}$ for $l = 1, 2, \ldots$. By (1.2) and (2.2) we have

$$a_i \leq (1 - h_i \tilde{\omega}(\alpha))^{-1} (1 + L_Z(\alpha) \|w_i - w_{i-1}\|_X) a_{i-1},$$

and condition (ii) implies $a_i \leq (1 - h_i \tilde{\omega}(\alpha))^{-1}(1 + L_Z(\alpha) L h_i) a_{i-1}$ for $i \geq k+1$. The desired estimate is obtained by iterating this inequalities, and using condition (1.1) and the estimate $(1-t)^{-1} \leq e^{2t}$ for $t \in [0, 1/2]$. ∎

LEMMA 2.4: *Let $\alpha > 0$ and $\tau > 0$. If a sequence $\{(t_i, u_i)\}_{i=0}^{\infty}$ in $[0, \tau) \times Y_\alpha$ satisfies two conditions*

(i) $0 = t_0 < t_1 < \cdots < t_i < \cdots < \tau$ *and* $\lim_{i \to \infty} t_i = \tau$,
(ii) $(u_i - u_{i-1})/(t_i - t_{i-1}) = A(u_{i-1}) u_i$ *for* $i = 1, 2, \ldots$,

then we have the following two assertions:

(a) *For each $z \in Z$ and sufficiently large k, the limit*

$$\lim_{i \to \infty} \prod_{l=k+1}^{i} (I - (t_l - t_{l-1}) A(u_{l-1}))^{-1} z \quad \text{exists in } Z.$$

(b) *The sequence $\{u_i\}$ converges in Y as $i \to \infty$.*

Proof: Since $u_i \in Y_\alpha$ for $i \geq 0$ the sequence $\{u_i\}$ is bounded in Y, by condition (φ1). We apply (1.4) together with this fact to the inequality obtained by condition (ii) that $\|u_i - u_{i-1}\|_X \leq (t_i - t_{i-1}) \|A(u_{i-1})\|_{Y,X} \|u_i\|_Y$ for $i \geq 1$. This yields that assumption (ii) of Lemma 2.3 is satisfied with $w_l = u_l$ and $L = M_A(\alpha) \sup\{\|u_i\|_Y : i \geq 0\}$. Since $t_l - t_{l-1} \to 0$ as $l \to \infty$ there is an integer $k_0 \geq 0$ such that $(t_l - t_{l-1}) \overline{\omega}(\alpha) \leq 1/2$ for $l \geq k_0 + 1$, where $\overline{\omega}(\alpha)$ is defined as in Remark 1.1. To prove assertion (a), let $k \geq k_0$. We have by Lemma 2.3

$$\left\| \prod_{l=k+1}^{i} (I - h_l A(u_{l-1}))^{-1} \right\|_{Z,Z} \leq M$$

for $i \geq k$, where $M = M(\alpha)\exp((2\omega(\alpha) + L_Z(\alpha)L)\tau)$ and $h_l = t_l - t_{l-1}$ for $l \geq 1$. A simple computation gives

$$\begin{aligned}(2.3)\qquad &\prod_{l=k+1}^{i} (I - h_l A(u_{l-1}))^{-1} y - \prod_{l=k+1}^{j} (I - h_l A(u_{l-1}))^{-1} y \\ &= \sum_{p=j+1}^{i} h_p A(u_{p-1}) \prod_{l=k+1}^{p} (I - h_l A(u_{l-1}))^{-1} y\end{aligned}$$

for $y \in Y$ and $i \geq j \geq k$. By (1.7), we apply Lemma 2.3 to the family $\{A(w) + B(w)\}$ and use the relation (1.8). This yields

$$\left\| \prod_{l=k+1}^{i} (I - h_l A(u_{l-1}))^{-1} \right\|_{Y,Y} \leq \overline{M}$$

for $i \geq k$, where $\overline{M} = M_Z(\alpha)^2 \|S\|_{Y,Z} \|S^{-1}\|_{Z,Y} \exp((2\overline{\omega}(\alpha) + L_Z(\alpha)L)\tau)$. By this fact the right-hand side of (2.3) is estimated by $(t_i - t_j) M_A(\alpha) \overline{M} \|y\|_Y$. Assertion (a) follows readily from the Banach–Steinhaus theorem.

We prove assertion (b). To do so, let $k \geq k_0$. By condition (ii) and assumption (A2) we have $(Su_i - Su_{i-1})/h_i = (A(u_{i-1}) + B(u_{i-1}))Su_i$; hence $Su_i = (I - h_i A(u_{i-1}))^{-1}(Su_{i-1} + h_i B(u_{i-1}) Su_i)$ for $i \geq k$. It is proved inductively that

$$\begin{aligned}(2.4)\qquad Su_i = &\prod_{l=k+1}^{i} (I - h_l A(u_{l-1}))^{-1} Su_k \\ &+ \sum_{l=k+1}^{i} \left(\prod_{p=l}^{i} (I - h_p A(u_{p-1}))^{-1} \right) h_l B(u_{l-1}) Su_l\end{aligned}$$

for $i \geq k$. We use this identity to represent the difference between Su_i and Su_j, and estimate it in Z by using (1.6). This yields

$$\begin{aligned}\|S(u_i - u_j)\|_Z \leq &\left\| \prod_{l=k+1}^{i} (I - h_l A(u_{l-1}))^{-1} Su_k - \prod_{l=k+1}^{j} (I - h_l A(u_{l-1}))^{-1} Su_k \right\|_Z \\ &+ ((t_i - t_k) + (t_j - t_k))K\end{aligned}$$

for $i \geq j \geq k$, where $K = MM_B(\alpha)\|S\|_{Y,Z} \sup\{\|u_i\|_Y : i \geq 0\}$. It follows by assertion (a) that $\limsup_{i,j\to\infty} \|S(u_i - u_j)\|_Z \leq 2(\tau - t_k)K$, and the right-hand side tends to zero as $k \to \infty$. This implies that assertion (b) is true. ∎

LEMMA 2.5: *Let $\alpha \geq 0$. If a sequence $\{u_i\}$ in Y_α converges to u in Y as $i \to \infty$, then there is an $h_0 > 0$ such that for $\lambda \in (0, h_0)$ and every sequence $\{\lambda_i\}$ which converges to λ as $i \to \infty$, we have*

$$\lim_{i\to\infty} (I - \lambda_i A(u_{i-1}))^{-1} u_{i-1} = (I - \lambda A(u))^{-1} u \quad \text{in } Y.$$

Proof: We choose $h_0 > 0$ so that $M_Z(\alpha)^2(1 - h_0\omega(\alpha))^{-1} h_0 M_B(\alpha) < 1$ and $h_0\overline{\omega}(\alpha) < 1$. Now, let $\lambda \in (0, h_0)$ and $\{\lambda_i\}$ any sequence with $\lim_{i\to\infty} \lambda_i = \lambda$. There is an integer $i_0 \geq 1$ such that $\lambda_i \in (0, h_0)$ for all $i \geq i_0$, and then for each $w \in Y_\alpha$, $(I - \lambda_i A(w))^{-1} \in B(Y)$ exists by Remark 1.1. Let $i \geq i_0$. If $\mathfrak{A}$ is a closed linear operator in Z then the resolvent $(\mu I - \mathfrak{A})^{-1}$ is analytic with respect to μ in the resolvent set. It follows from condition (A1) and (1.7) that for each $y \in Y$, $(I - \mu A(u))^{-1} y$ and $(I - \mu(A(u) + B(u)))^{-1} S y$ is continuous on $(0, h_0)$. These facts and (1.8) together imply that for each $y \in Y$,

$$\lim_{i\to\infty} (I - \lambda_i A(u))^{-1} y = (I - \lambda A(u))^{-1} y \quad \text{in } Y. \tag{2.5}$$

It remains to show $\lim_{i\to\infty} \|(I - \lambda_i A(u_{i-1}))^{-1} u_{i-1} - (I - \lambda_i A(u))^{-1} u\|_Y = 0$. By (A1) and (1.1) we have

$$\|(I - \lambda_i A(w))^{-1}\|_{Z,Z} \leq M_Z(\alpha)^2 (1 - \lambda_i \omega(\alpha))^{-1} \tag{2.6}$$

for $w \in Y_\alpha$. Since

$$\begin{aligned}(I - \lambda_i A(u_{i-1}))^{-1} y &- (I - \lambda_i A(u))^{-1} y \\ &= \lambda_i (I - \lambda_i A(u_{i-1}))^{-1} (A(u_{i-1}) - A(u))(I - \lambda_i A(u))^{-1} y,\end{aligned}$$

we have

$$\begin{aligned}\|(I - \lambda_i A(u_{i-1}))^{-1} y &- (I - \lambda_i A(u))^{-1} y\|_Z \\ &\leq \lambda_i M_Z(\alpha)^2 (1 - \lambda_i \omega(\alpha))^{-1} L_A(\alpha) \|u_{i-1} - u\|_Z \|(I - \lambda_i A(u))^{-1} y\|_Y\end{aligned}$$

for $y \in Y$, and the right-hand side tends to zero as $i \to \infty$. It follows from the Banach–Steinhaus theorem that for each $z \in Z$,

$$\lim_{i\to\infty} \|(I - \lambda_i A(u_{i-1}))^{-1} z - (I - \lambda_i A(u))^{-1} z\|_Z = 0.$$

Now, put $v_i = (I - \lambda_i A(u_{i-1}))^{-1} u_{i-1}$ and $\tilde{v}_i = (I - \lambda_i A(u))^{-1} u$. We use assumption (A2) to find

$$S v_i = (I - \lambda_i A(u_{i-1}))^{-1} (S u_{i-1} + \lambda_i B(u_{i-1}) S v_i) \tag{2.7}$$

and

$$S\tilde{v}_i = (I - \lambda_i A(u))^{-1}(Su + \lambda_i B(u)S\tilde{v}_i). \tag{2.8}$$

Subtracting (2.8) from (2.7) and estimating the resultant equality by (1.6) and (2.6), we have

$$\begin{aligned} \|S(v_i - \tilde{v}_i)\|_Z \leq & \|((I - \lambda_i A(u_{i-1}))^{-1} - (I - \lambda_i A(u))^{-1})(Su + \lambda_i B(u)S\tilde{v}_i)\|_Z \\ & + M_Z(\alpha)^2(1 - \lambda_i\omega(\alpha))^{-1}\{\|S(u_{i-1} - u)\|_Z \\ & \quad + \lambda_i\|(B(u_{i-1}) - B(u))S\tilde{v}_i\|_Z + \lambda_i M_B(\alpha)\|S(v_i - \tilde{v}_i)\|_Z\}, \end{aligned}$$

and the first term on the right-hand side vanishes as $i \to \infty$, by (2.5) and what we have shown above. We set $\delta = \limsup_{i\to\infty} \|S(v_i - \tilde{v}_i)\|_Z$ and take the limit as $i \to \infty$. This yields $\delta \leq M_Z(\alpha)^2(1 - \lambda\omega(\alpha))^{-1}\lambda M_B(\alpha)\delta$. Here we have used (1.3). By the choice of h_0 we have $\delta = 0$; hence the sequence $\{v_i\}$ converges to $(I - \lambda A(u))^{-1}u$ in Y as $i \to \infty$. ∎

Proof of Theorem 2.1: Let $\varepsilon > 0$ and $u_0 \in Y$. Let $i \geq 1$, and assume that a sequence $\{(t_l, u_l)\}_{l=0}^{i-1}$ in $[0, \tau_\varepsilon(\varphi(u_0))) \times Y$ has been chosen so that (i) through (iv) may hold for $0 \leq l \leq i-1$. We then denote by $\overline{h}_i$ the supremum of all $h \in [0, \varepsilon]$ such that

$$\begin{cases} t_{i-1} + h < \tau_\varepsilon(\varphi(u_0)), \\ \varphi((I - hA(u_{i-1}))^{-1}u_{i-1}) \leq m_\varepsilon(h; \varphi(u_{i-1})). \end{cases}$$

By Lemma 2.2 we have $\overline{h}_i > 0$. This fact enables us to choose $h_i \in (0, \varepsilon]$ so that $\overline{h}_i/2 < h_i$, $t_{i-1} + h_i < \tau_\varepsilon(\varphi(u_0))$ and

$$\varphi((I - h_i A(u_{i-1}))^{-1}u_{i-1}) \leq m_\varepsilon(h_i; \varphi(u_{i-1})). \tag{2.9}$$

Now, we put $t_i = t_{i-1} + h_i$ and $u_i = (I - h_i A(u_{i-1}))^{-1}u_{i-1}$. Clearly, condition (i) through (iii) are satisfied. To show that condition (iv) is true in the case of i, we note by (iii) of Proposition 1.2 that

$$\tau_\varepsilon(m_\varepsilon(t_{i-1}; \varphi(u_0))) = \tau_\varepsilon(\varphi(u_0)) - t_{i-1} > h_i,$$

since $t_{i-1} \in [0, \tau_\varepsilon(\varphi(u_0)))$. Using the hypothesis of induction that $\varphi(u_{i-1}) \leq m_\varepsilon(t_{i-1}; \varphi(u_0))$, we have by (i) of Proposition 1.2

$$m_\varepsilon(h_i; \varphi(u_{i-1})) \leq m_\varepsilon(h_i; m_\varepsilon(t_{i-1}; \varphi(u_0)))$$

and the right-hand side is equal to $m_\varepsilon(h_i + t_{i-1}; \varphi(u_0))$ by (iii) of Proposition 1.2 again. The claim that condition (iv) holds follows by combining the fact above and the inequality (2.9).

It remains to prove condition (v). For the purpose of an indirect proof, it is assumed that $\overline{t} := \lim_{i\to\infty} t_i < \tau_\varepsilon(\varphi(u_0))$. We then have

$$\overline{\alpha} := \sup\{m_\varepsilon(t; \varphi(u_0)) \colon t \in [0, \overline{t}]\} < \infty.$$

Condition (iv) implies $u_i \in Y_{\overline{\alpha}}$ for $i \geq 1$. It follows from Lemma 2.4 that the sequence $\{u_i\}$ converges in Y as $i \to \infty$. Now, put $\overline{u} = \lim_{i\to\infty} u_i \in Y$. By Lemma 2.5 there is an $h_0 > 0$ such that for $\lambda \in (0, h_0)$ and every sequence $\{\lambda_i\}$ with $\lim_{i\to\infty} \lambda_i = \lambda$, $\lim_{i\to\infty}(I - \lambda_i A(u_{i-1}))^{-1}u_{i-1} = (I - \lambda A(\overline{u}))^{-1}\overline{u}$ in Y. We choose $h \in (0, (\varepsilon \wedge h_0)/2]$ such that

$$\begin{cases} \overline{t} + h < \tau_\varepsilon(\varphi(u_0)), \\ \varphi((I - hA(\overline{u}))^{-1}\overline{u}) \leq m_{\varepsilon/2}(h; \varphi(\overline{u})). \end{cases}$$

Here we have used Lemma 2.2. Set $\gamma_i = \overline{t} + h - t_{i-1}$ for $i \geq 1$. Since $\overline{h}_i < 2h_i = 2(t_i - t_{i-1}) \to 0$ and $\gamma_i \to h$ as $i \to \infty$, there is an integer $i_0 \geq 1$ such that $\overline{h}_i < \gamma_i \leq \varepsilon$ for all $i \geq i_0$. Clearly, $t_{i-1} + \gamma_i < \tau_\varepsilon(\varphi(u_0))$ for all $i \geq 1$. By the definition of $\overline{h}_i$ we have

$$\varphi((I - \gamma_i A(u_{i-1}))^{-1}u_{i-1}) > m_\varepsilon(\gamma_i; \varphi(u_{i-1})) \tag{2.10}$$

for all $i \geq i_0$. Condition (iv) implies

$$\tau_\varepsilon(\varphi(u_k)) \geq \tau_\varepsilon(m_\varepsilon(t_k; \varphi(u_0)) = \tau_\varepsilon(\varphi(u_0)) - t_k > \overline{t} + h - t_k > t_j - t_k$$

for $j \geq k \geq 0$, by Proposition 1.2. By (2.9) we have inductively $\varphi(u_j) \leq m_\varepsilon(t_j - t_k; \varphi(u_k))$ for $j \geq k \geq 0$, which gives $\varphi(\overline{u}) \leq m_\varepsilon(\overline{t} - t_{i-1}; \varphi(u_{i-1}))$ for $i \geq i_0$; hence we have by Proposition 1.2,

$$m_\varepsilon(h; \varphi(\overline{u})) \leq m_\varepsilon(\gamma_i; \varphi(u_{i-1})) \tag{2.11}$$

for $i \geq i_0$. Combining (2.10) and (2.11), and taking the limit as $i \to \infty$ we find

$$\varphi((I - hA(\overline{u}))^{-1}\overline{u}) \geq m_\varepsilon(h; \varphi(\overline{u})),$$

which is a contradiction to the choice of h, by (iv) of Proposition 1.2. It is concluded that condition (v) holds. ∎

3. Convergence of approximate solutions and proof of main theorem

In this section we investigate the convergence of "semi-implicit" discrete approximate solutions of (QE) and prove the main theorem.

THEOREM 3.1: *Let $u_0 \in Y$ and $T > 0$. Suppose that for each $\varepsilon > 0$, there exist a sequence $\{t_i^\varepsilon\}_{i=0}^{N_\varepsilon}$ of nonnegative numbers and a sequence $\{u_i^\varepsilon\}_{i=0}^{N_\varepsilon}$ in Y such that they satisfy the following conditions:*

(i) $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_i^\varepsilon < \cdots$ *and* $T \le t_{N_\varepsilon}^\varepsilon < T + \varepsilon$;
(ii) $t_i^\varepsilon - t_{i-1}^\varepsilon \le \varepsilon$ *for* $i = 1, 2, \ldots, N_\varepsilon$;
(iii) $(u_i^\varepsilon - u_{i-1}^\varepsilon)/(t_i^\varepsilon - t_{i-1}^\varepsilon) = A(u_{i-1}^\varepsilon)u_i^\varepsilon$ *for* $i = 1, 2, \ldots, N_\varepsilon$, *where* $u_0^\varepsilon = u_0$.

If we define a simple function $u^\varepsilon \colon [0, T] \to Y$ *by*

$$u^\varepsilon(t) = \begin{cases} u_0 & \text{for } t = 0, \\ u_i^\varepsilon & \text{for } t \in (t_{i-1}^\varepsilon, t_i^\varepsilon] \cap [0, T] \text{ and } i = 1, 2, \ldots, N_\varepsilon, \end{cases}$$

then the following statements are equivalent:

(a) $\sup\{\|u^\varepsilon(t)\|_Y : t \in [0, T]\}$ *is bounded as* $\varepsilon \downarrow 0$.
(b) *There is a classical solution* u *to* (QE) *on* $[0, T]$ *such that*

$$\lim_{\varepsilon \downarrow 0} (\sup\{\|u^\varepsilon(t) - u(t)\|_Y : t \in [0, T]\}) = 0. \tag{3.1}$$

Proof: It is obvious that (b) implies (a). We prove the implication "(a) $\Rightarrow$ (b)". If there exists $u \in C([0, T] \colon Y)$ satisfying (3.1), then we see that u is a classical solution to (QE) on $[0, T]$, by letting $\varepsilon \downarrow 0$ in the equality

$$u^\varepsilon(t) - u_0 = \sum_{l=1}^{i} \int_{t_{l-1}^\varepsilon}^{t_l^\varepsilon} A(u^\varepsilon(t_{l-1}^\varepsilon))u^\varepsilon(r)\, dr$$

for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$ which follows readily from (iii). By this fact it suffices to prove that there exists $u \in C([0, T] \colon Y)$ satisfying (3.1). The proof will be divided into a sequence of lemmas. Now, we assume (a), and so there exists an $\varepsilon_0 > 0$ such that

$$r_0 = \sup\{\|u_i^\varepsilon\|_Y : 0 \le i \le N_\varepsilon \text{ and } \varepsilon \in (0, \varepsilon_0]\} < \infty.$$

Condition ($\varphi 2$) implies

$$\alpha_0 = \sup\{\varphi(u_i^\varepsilon) : 0 \le i \le N_\varepsilon \text{ and } \varepsilon \in (0, \varepsilon_0]\} < \infty.$$

Let $\overline{\omega}(\alpha_0)$ be the nonnegative number defined as in Remark 1.1, and set $\Delta = \{(t,s)\colon 0 \le s \le t \le T\}$. For each $\varepsilon \in (0,\varepsilon_0]$ with $\varepsilon\overline{\omega}(\alpha_0) \le 1/2$, we introduce a family $\{U_\varepsilon(t,s)\colon (t,s) \in \Delta\}$ in $B(Z)$ defined by

$$U_\varepsilon(t,s) = \prod_{l=p+1}^{i} (I - h_l^\varepsilon A(u_{l-1}^\varepsilon))^{-1}$$

for $s \in (t_{p-1}^\varepsilon, t_p^\varepsilon] \cap [0,T]$ and $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon] \cap [0,T]$. Here and subsequently, t_{-1}^ε is defined by $t_{-1}^\varepsilon = -\infty$ for convenience, and we write for simplicity $h_l^\varepsilon = t_l^\varepsilon - t_{l-1}^\varepsilon$ for $1 \le l \le N_\varepsilon$.

We start with the following lemma on the uniform boundedness of $U_\varepsilon(t,s)$ in $B(Z)$ and $B(Y)$ norm.

LEMMA 3.2: *For each $\varepsilon \in (0,\varepsilon_0]$ with $\varepsilon\overline{\omega}(\alpha_0) \le 1/2$ we have*

(i) $\|U_\varepsilon(t,s)\|_{Z,Z} \le M(\alpha_0)\exp(\beta_0(t_i^\varepsilon - t_k^\varepsilon))$,

(ii) $\|U_\varepsilon(t,s)\|_{Y,Y} \le \overline{M}(\alpha_0)\exp(\overline{\beta}_0(t_i^\varepsilon - t_k^\varepsilon))$

for $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon] \cap [0,T]$ and $s \in (t_{k-1}^\varepsilon, t_k^\varepsilon] \cap [0,T]$, where

$$\overline{M}(\alpha_0) = \|S\|_{Y,Z}\|S^{-1}\|_{Z,Y}M(\alpha_0),$$
$$\beta_0 = 2\omega(\alpha_0) + L_Z(\alpha_0)M_A(\alpha_0)r_0, \quad \text{and} \quad \overline{\beta}_0 = 2\overline{\omega}(\alpha_0) + L_Z(\alpha_0)M_A(\alpha_0)r_0.$$

Proof: Let $\varepsilon \in (0,\varepsilon_0]$ be such that $\varepsilon\overline{\omega}(\alpha_0) \le 1/2$ which implies $(t_i^\varepsilon - t_{i-1}^\varepsilon)\overline{\omega}(\alpha_0) \le 1/2$ for $i = 1,2,\dots,N_\varepsilon$. By condition (iii) of Theorem 3.1 we have

$$\text{(3.2)} \qquad \|u_i^\varepsilon - u_{i-1}^\varepsilon\|_X \le (t_i^\varepsilon - t_{i-1}^\varepsilon)\|A(u_{i-1}^\varepsilon)\|_{Y,X}\|u_i^\varepsilon\|_Y \le M_A(\alpha_0)r_0(t_i^\varepsilon - t_{i-1}^\varepsilon)$$

for $i = 1,2,\dots,N_\varepsilon$. Assertion (i) is a direct consequence of Lemma 2.3. By (1.7) we apply Lemma 2.3 again to the family $\{A(w) + B(w)\}$ and use the relation (1.8). This proves that assertion (ii) is true. ∎

Let $\lambda,\mu \in (0,\varepsilon_0]$ be such that $(\lambda \vee \mu)\overline{\omega}(\alpha_0) \le 1/2$ and $y \in Y$. For $0 \le p \le N_\lambda$ and $0 \le q \le N_\mu$, we shall estimate the norm in Z of the difference between

$$\text{(3.3)} \qquad z_i^\lambda = \prod_{l=p+1}^{i} (I - h_l^\lambda A(u_{l-1}^\lambda))^{-1}y \quad \text{for } i = p, p+1, \dots, N_\lambda,$$

and

$$\text{(3.4)} \qquad \hat{z}_j^\mu = \prod_{l=q+1}^{j} (I - h_l^\mu A(u_{l-1}^\mu))^{-1}y \quad \text{for } j = q, q+1, \dots, N_\mu.$$

It is convenient to employ the following notations:

$$a_{i,j}^{\lambda,\mu} = \|z_i^\lambda - \hat{z}_j^\mu\|_{(u_i^\lambda)} \vee \|z_i^\lambda - \hat{z}_j^\mu\|_{(u_j^\mu)} \quad \text{for } p \le i \le N_\lambda \text{ and } q \le j \le N_\mu$$

and

$$b_{i,j}^{\lambda,\mu} = \|u_i^\lambda - u_j^\mu\|_{(u_i^\lambda)} \vee \|u_i^\lambda - u_j^\mu\|_{(u_j^\mu)} \quad \text{for } 0 \le i \le N_\lambda \text{ and } 0 \le j \le N_\mu.$$

By (ii) of Lemma 3.2 there exists $C(\|y\|_Y) > 0$ such that

$$\sup\{\|z_i^\lambda\|_Y : p \le i \le N_\lambda\} \vee \sup\{\|\hat{z}_j^\mu\|_Y : q \le j \le N_\mu\} \le C(\|y\|_Y)$$

for $0 \le p \le N_\lambda$ and $0 \le q \le N_\mu$. The following fundamental inequalities will be used for the comparison between z_i^λ and $\hat{z}_j^\mu$ by induction on (i, j).

LEMMA 3.3: (i) *The inequality*

$$\begin{aligned} &\left(1 - \frac{h_i^\lambda h_j^\mu}{h_i^\lambda + h_j^\mu}\omega\right) a_{i,j}^{\lambda,\mu} \\ &\le \frac{h_j^\mu}{h_i^\lambda + h_j^\mu}(1 + Lh_i^\lambda)a_{i-1,j}^{\lambda,\mu} + \frac{h_i^\lambda}{h_i^\lambda + h_j^\mu}(1 + Lh_j^\mu)a_{i,j-1}^{\lambda,\mu} \\ &\quad + \frac{h_i^\lambda h_j^\mu}{h_i^\lambda + h_j^\mu}\{K(\|y\|_Y)(\lambda + \mu) + M(\|y\|_Y)(b_{i-1,j}^{\lambda,\mu} + b_{i,j-1}^{\lambda,\mu})\} \end{aligned} \tag{3.5}$$

holds for $p + 1 \le i \le N_\lambda$ *and* $q + 1 \le j \le N_\mu$, *where*

$$\begin{aligned} &\omega = \omega(\alpha_0), \quad L = (L_Z(\alpha_0)M_A(\alpha_0) \vee M_Z(\alpha_0)^2 L_A(\alpha))r_0, \\ &K(\|y\|_Y) = M_Z(\alpha_0)L_A(\alpha_0)M_A(\alpha_0)r_0 C(\|y\|_Y), \text{ and} \\ &M(\|y\|_Y) = M_Z(\alpha_0)^2 L_A(\alpha_0)C(\|y\|_Y). \end{aligned}$$

(ii) $a_{i,j}^{\lambda,\mu} \le |(t_i^\lambda - t_p^\lambda) - (t_j^\mu - t_q^\mu)|N(\|y\|_Y)$ *for* $i = p$ *or* $j = q$, *where* $N(\|y\|_Y) = M_Z(\alpha_0)M_A(\alpha_0)C(\|y\|_Y)$.

Proof: Let $p + 1 \le i \le N_\lambda$ and $q + 1 \le j \le N_\mu$. From the definition of z_i^λ and $\hat{z}_j^\mu$ it follows readily that

$$(z_i^\lambda - z_{i-1}^\lambda)/h_i^\lambda = A(u_{i-1}^\lambda)z_i^\lambda = A(u_i^\lambda)z_i^\lambda + (A(u_{i-1}^\lambda) - A(u_i^\lambda))z_i^\lambda$$

and

$$(\hat{z}_j^\mu - \hat{z}_{j-1}^\mu)/h_j^\mu = A(u_{j-1}^\mu)\hat{z}_j^\mu = A(u_i^\lambda)\hat{z}_j^\mu + (A(u_{j-1}^\mu) - A(u_i^\lambda))\hat{z}_j^\mu.$$

Using the assumption that $A(u_i^\lambda) \in G(Z_{(u_i^\lambda)}, 1, \omega)$ we find

$$(3.6) \qquad \begin{aligned} &(\|z_i^\lambda - \hat{z}_j^\mu\|_{(u_i^\lambda)} - \|z_{i-1}^\lambda - \hat{z}_j^\mu\|_{(u_i^\lambda)})/h_i^\lambda \\ &\quad + (\|z_i^\lambda - \hat{z}_j^\mu\|_{(u_i^\lambda)} - \|z_i^\lambda - \hat{z}_{j-1}^\mu\|_{(u_i^\lambda)})/h_j^\mu \\ &\qquad \le \omega\|z_i^\lambda - \hat{z}_j^\mu\|_{(u_i^\lambda)} + \|(A(u_{i-1}^\lambda)-A(u_i^\lambda))z_i^\lambda\|_{(u_i^\lambda)} \\ &\qquad\quad + \|(A(u_{j-1}^\mu)-A(u_i^\lambda))\hat{z}_j^\mu\|_{(u_i^\lambda)}. \end{aligned}$$

By (1.2) and (3.2) we have

$$\begin{aligned} \|z_{i-1}^\lambda - \hat{z}_j^\mu\|_{(u_i^\lambda)} &\le \|z_{i-1}^\lambda - \hat{z}_j^\mu\|_{(u_{i-1}^\lambda)}(1 + L_Z(\alpha_0)M_A(\alpha_0)r_0h_i^\lambda) \\ &\le a_{i-1,j}^{\lambda,\mu}(1 + Lh_i^\lambda). \end{aligned}$$

We find by (3.2) again

$$\begin{aligned} \|(A(u_{i-1}^\lambda) - A(u_i^\lambda))z_i^\lambda\|_{(u_i^\lambda)} &\le M_Z(\alpha_0)\|(A(u_{i-1}^\lambda) - A(u_i^\lambda))z_i^\lambda\|_Z \\ \le M_Z(\alpha_0)L_A(\alpha_0)\|u_{i-1}^\lambda - u_i^\lambda\|_Z\|z_i^\lambda\|_Y &\le K(\|y\|_Y)\lambda. \end{aligned}$$

Similarly we see that the last term on the right-hand side of (3.6) is bounded by $M(\|y\|_Y)b_{i,j-1}^{\lambda,\mu}$. Manipulating these inequalities we have

$$(3.7) \qquad \begin{aligned} &\left(1 - \frac{h_i^\lambda h_j^\mu}{h_i^\lambda + h_j^\mu}\omega\right)\|z_i^\lambda - \hat{z}_j^\mu\|_{(u_i^\lambda)} \\ &\quad \le \frac{h_j^\mu}{h_i^\lambda + h_j^\mu}(1 + Lh_i^\lambda)a_{i-1,j}^{\lambda,\mu} + \frac{h_i^\lambda}{h_i^\lambda + h_j^\mu}a_{i,j-1}^{\lambda,\mu} \\ &\qquad + \frac{h_i^\lambda h_j^\mu}{h_i^\lambda + h_j^\mu}(K(\|y\|_Y)\lambda + M(\|y\|_Y)b_{i,j-1}^{\lambda,\mu}). \end{aligned}$$

The desired inequality (i) is obtained by applying this argument with (p, i, λ) and (q, j, μ) interchanged, and combining the resultant inequality and (3.7).

We now turn to the proof of (ii). Since $z_i^\lambda - z_{i-1}^\lambda = h_i^\lambda A(u_{i-1}^\lambda)z_i^\lambda$ we have $\|z_i^\lambda - z_{i-1}^\lambda\|_Z \le h_i^\lambda M_A(\alpha_0)C(\|y\|_Y)$ for $p+1 \le i \le N_\lambda$. By (1.1) we have

$$a_{i,q}^{\lambda,\mu} \le M_Z(\alpha_0)\|z_i^\lambda - y\|_Z \le M_Z(\alpha_0)\|z_i^\lambda - z_p^\lambda\|_Z$$

for $p \le i \le N_\lambda$. It follows that $a_{i,q}^{\lambda,\mu} \le (t_i^\lambda - t_p^\lambda)N(\|y\|_Y)$ for $p \le i \le N_\lambda$. Similarly, we have $a_{p,j}^{\lambda,\mu} \le (t_j^\mu - t_q^\mu)N(\|y\|_Y)$ for $q \le j \le N_\mu$. ∎

The following two lemmas are needed to show that the limit $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t)$ exists in the "good" subspace Y of Z, uniformly on $[0, T]$.

LEMMA 3.4: *For $0 \le s \le t \le T$ we have*

$$Su^\varepsilon(t) = U_\varepsilon(t,s)Su^\varepsilon(s) + \sum_{l=k+1}^{i} \int_{t_{l-1}^\varepsilon}^{t_l^\varepsilon} U_\varepsilon(t, t_{l-1}^\varepsilon)B(u^\varepsilon(t_{l-1}^\varepsilon))Su^\varepsilon(r)\,dr,$$

where i and k are nonnegative integers such that $t \in (t_{i-1}^\varepsilon, t_i^\varepsilon]$ and $s \in (t_{k-1}^\varepsilon, t_k^\varepsilon]$.

Lemma 3.4 is readily proved by taking account of (2.4). ∎

LEMMA 3.5: *For each $z \in Z$ we have*

(i) *The limit $U(t,s)z := \lim_{\varepsilon\downarrow 0} U_\varepsilon(t,s)z$ exists in Z uniformly on Δ;*
(ii) *the function $(t,s) \to U(t,s)z$ is continuous in Z on Δ;*
(iii) *$U(t,t)z = z$, and $U(t,s)z = U(t,r)U(r,s)z$ for $(t,r),(r,s) \in \Delta$.*

Proof: By (i) of Lemma 3.2, it suffices to prove the lemma for all $y \in Y$. For this purpose, let $\lambda, \mu \in (0, \varepsilon_0]$ be such that $(\lambda \vee \mu)\overline{\omega}(\alpha_0) \le 1/2$ and $y \in Y$. Let us define z_i^λ and $\hat{z}_j^\mu$ by (3.3) and (3.4) respectively, and then the inequalities (i) and (ii) of Lemma 3.3 hold. We begin by showing the estimate on $b_{i,j}^{\lambda,\mu}$. For simplicity in notation we write

$$\gamma_{i,j}^{\lambda,\mu} = \prod_{k=1}^{i}(1-\omega h_k^\lambda)(1+2Lh_k^\lambda)^{-1} \cdot \prod_{k=1}^{j}(1-\omega h_k^\mu)(1+2Lh_k^\mu)^{-1}$$

for $0 \le i \le N_\lambda$ and $0 \le j \le N_\mu$. It should be noted here that $z_i^\lambda = u_i^\lambda$ and $\hat{z}_j^\mu = u_j^\mu$ if $p = q = 0$ and $y = u_0$. Setting $p = q = 0$ and $y = u_0$ in the inequalities (i) and (ii) of Lemma 3.3, and multiplying the resultant inequalities on only $b_{i,j}^{\lambda,\mu}$, that is, the inequalities with $C(\|y\|_Y)$ replaced by r_0, we find

$$b_{i,j}^{\lambda,\mu} \le N|t_i^\lambda - t_j^\mu| \quad \text{for } i = 0 \text{ or } j = 0$$

and

$$\gamma_{i,j}^{\lambda,\mu} b_{i,j}^{\lambda,\mu} \le \frac{h_j^\mu}{h_i^\lambda + h_j^\mu}\gamma_{i-1,j}^{\lambda,\mu} b_{i-1,j}^{\lambda,\mu} + \frac{h_i^\lambda}{h_i^\lambda + h_j^\mu}\gamma_{i,j-1}^{\lambda,\mu} b_{i,j-1}^{\lambda,\mu} + \frac{h_i^\lambda h_j^\mu}{h_i^\lambda + h_j^\mu}K(\lambda+\mu)$$

for $1 \le i \le N_\lambda$ and $1 \le j \le N_\mu$, where $N = M_Z(\alpha_0)M_A(\alpha_0)r_0$ and $K = M_Z(\alpha_0)L_A(\alpha_0)M_A(\alpha_0)r_0^2$. On the other hand, we deduce from Schwarz's inequality that a sequence $\{\beta_{i,j}^{\lambda,\mu}\}$ of nonnegative numbers defined by

$$\beta_{i,j}^{\lambda,\mu} = N((t_i^\lambda - t_j^\mu)^2 + \lambda t_i^\lambda + \mu t_j^\mu)^{1/2} + K(\lambda t_i^\lambda + \mu t_j^\mu)$$

for $0 \le i \le N_\lambda$ and $0 \le j \le N_\mu$ satisfies the inequality

$$\beta_{i,j}^{\lambda,\mu} \ge \frac{h_j^\mu}{h_i^\lambda + h_j^\mu}\beta_{i-1,j}^{\lambda,\mu} + \frac{h_i^\lambda}{h_i^\lambda + h_j^\mu}\beta_{i,j-1}^{\lambda,\mu} + \frac{h_i^\lambda h_j^\mu}{h_i^\lambda + h_j^\mu}K(\lambda+\mu) \tag{3.8}$$

for $1 \le i \le N_\lambda$ and $1 \le j \le N_\mu$. One verifies inductively

$$\gamma_{i,j}^{\lambda,\mu} b_{i,j}^{\lambda,\mu} \le \beta_{i,j}^{\lambda,\mu} \tag{3.9}$$

for $0 \le i \le N_\lambda$ and $0 \le j \le N_\mu$. (See also Kobayashi [7].)

We now turn to the estimate on $a_{i,j}^{\lambda,\mu}$. It is necessary for us to rewrite (3.5). To do so, we use for simplicity the notation

$$\omega_{i,j}^{\lambda,\mu} = \prod_{k=p+1}^{i} (1-\omega h_k^\lambda)(1+Lh_k^\lambda)^{-1} \cdot \prod_{k=q+1}^{j} (1-\omega h_k^\mu)(1+Lh_k^\mu)^{-1}$$

for $p \le i \le N_\lambda$ and $q \le j \le n_\mu$. Multiplying (3.5) by $\omega_{i,j}^{\lambda,\mu}\gamma_{i,j}^{\lambda,\mu}$ we find

$$\begin{aligned}\gamma_{i,j}^{\lambda,\mu}\omega_{i,j}^{\lambda,\mu}a_{i,j}^{\lambda,\mu} \le{}& \frac{h_j^\mu}{h_i^\lambda + h_j^\mu}\gamma_{i-1,j}^{\lambda,\mu}\omega_{i-1,j}^{\lambda,\mu}a_{i-1,j}^{\lambda,\mu} + \frac{h_i^\lambda}{h_i^\lambda + h_j^\mu}\gamma_{i,j-1}^{\lambda,\mu}\omega_{i,j-1}^{\lambda,\mu}a_{i,j-1}^{\lambda,\mu}\\ &+ \frac{h_i^\lambda h_j^\mu}{h_i^\lambda + h_j^\mu}K(\|y\|_Y)(\lambda+\mu)\\ &+ \frac{h_i^\lambda h_j^\mu}{h_i^\lambda + h_j^\mu}M(\|y\|_Y)(\gamma_{i-1,j}^{\lambda,\mu}b_{i-1,j}^{\lambda,\mu} + \gamma_{i,j-1}^{\lambda,\mu}b_{i,j-1}^{\lambda,\mu})\end{aligned}$$

for $p+1 \le i \le N_\lambda$ and $q+1 \le j \le N_\mu$. Using (3.8) and (3.9) we have by induction on (i,j),

$$\begin{aligned}\gamma_{i,j}^{\lambda,\mu}\omega_{i,j}^{\lambda,\mu}a_{i,j}^{\lambda,\mu} \le{}& N(\|y\|_Y)\{((t_i^\lambda - t_p^\lambda) - (t_j^\mu - t_q^\mu))^2 + \lambda(t_i^\lambda - t_p^\lambda) + \mu(t_j^\mu - t_q^\mu)\}^{1/2}\\ &+ K(\|y\|_Y)(\lambda(t_i^\lambda - t_p^\lambda) + \mu(t_j^\mu - t_q^\mu))\\ &+ M(\|y\|_Y)\beta_{i,j}^{\lambda,\mu}((t_i^\lambda - t_p^\lambda) + (t_j^\mu - t_q^\mu))\end{aligned}$$

for $p \le i \le N_\lambda$ and $q \le j \le N_\mu$. Assertions (i) and (ii) are then proved by standard arguments. Assertion (iii) follows readily from the relation $U_\varepsilon(t,s) = U_\varepsilon(t,r)U_\varepsilon(r,s)$ for $(t,r),(r,s) \in \Delta$. ∎

The proof of the implication "(a) ⇒ (b)" is finally complete by the following lemma.

LEMMA 3.6: *There exists $u \in C([0,T]: Y)$ satisfying* (3.1).

Proof: We begin by showing that

$$\lim_{\varepsilon\downarrow 0}(\sup\{\|u^\varepsilon(t)-u(t)\|_Y : t\in[0,\tau]\}) = 0, \tag{3.10}$$

if there exists $u \in C([0,\tau]: Y)$ such that

$$Su(t) = U(t,0)Su_0 + \int_0^t U(t,r)B(u(r))Su(r)\,dr \tag{3.11}$$

for $t \in [0,\tau]$, where $\tau \in (0,T]$. To this end, let $t \in [0,\tau]$. Then there is an integer $i \geq 0$ such that $t \in (t^\varepsilon_{i-1}, t^\varepsilon_i]$. By Lemma 3.4 we have

$$Su^\varepsilon(t) = U_\varepsilon(t,0)Su_0 + \sum_{l=1}^{i}\int_{t^\varepsilon_{l-1}}^{t^\varepsilon_l} U_\varepsilon(t,t^\varepsilon_{l-1})B(u^\varepsilon(t^\varepsilon_{l-1}))Su^\varepsilon(r)\,dr. \tag{3.12}$$

By (i) of Lemma 3.5, we have $\lim_{\varepsilon\downarrow 0} U_\varepsilon(t,0)Su_0 = U(t,0)Su_0$ uniformly on $[0,\tau]$. We estimate $B(u^\varepsilon(t^\varepsilon_{l-1}))Su^\varepsilon(r) - B(u(r))Su(r)$ by dividing it into three terms

$$(B(u^\varepsilon(t^\varepsilon_{l-1})) - B(u(t^\varepsilon_{l-1})))Su^\varepsilon(r),\quad B(u(t^\varepsilon_{l-1}))(Su^\varepsilon(r)-Su(r)),\quad \text{and}$$
$$(B(u(t^\varepsilon_{l-1})) - B(u(r)))Su(r),$$

and then using (1.3) and (1.6). This yields

$$\begin{aligned}
&\|B(u^\varepsilon(t^\varepsilon_{l-1}))Su^\varepsilon(r) - B(u(r))Su(r)\|_Z \\
&\quad\leq L_B(\alpha_0\vee\alpha_1)\phi^\varepsilon(r)\|S\|_{Y,Z}r_0 + M_B(\alpha_1)\|S\|_{Y,Z}\phi^\varepsilon(r) \\
&\qquad + L_B(\alpha_1)\rho(\varepsilon)\|S\|_{Y,Z}r_1
\end{aligned}$$

for $r \in (t^\varepsilon_{l-1}, t^\varepsilon_l]$, where

$$r_1 = \sup\{\|u(t)\|_Y : t\in[0,\tau]\} \quad\text{and}\quad \alpha_1 = \sup\{\varphi(w) : \|w\|_Y \leq r_1\}.$$

Here two functions ϕ^ε and ρ are defined by

$$\phi^\varepsilon(t) = \sup\{\|u^\varepsilon(\eta)-u(\eta)\|_Y : \eta\in[0,t]\}$$

and

$$\rho(t) = \sup\{\|u(s)-u(\hat{s})\|_Y : |s-\hat{s}|\leq t\},$$

respectively. Subtracting (3.11) from (3.12) we find

$$\|u^\varepsilon(t) - u(t)\|_Y \le \lambda_\varepsilon + K_0 \int_0^t \phi^\varepsilon(r)\,dr$$

for $t \in [0,\tau]$, where K_0 is a positive constant and $\{\lambda_\varepsilon\}$ is a null sequence of positive numbers. We now set $\phi(t) = \limsup_{\varepsilon\downarrow 0} \phi^\varepsilon(t)$ for $t \in [0,\tau]$. By Lebesgue's convergence theorem we have $\phi(t) \le K_0 \int_0^t \phi(r)\,dr$ for $t \in [0,\tau]$. Application of Gronwall's inequality gives $\phi = 0$ on $[0,\tau]$, which proves the desired claim (3.10).

Now, we define $\overline{\tau}$ by the supremum $\tau \in [0,T]$ such that there exists $u \in C([0,\tau]\colon Y)$ satisfying (3.10). We shall show that there exists $u \in C([0,\overline{\tau}]\colon Y)$ such that (3.11) holds with τ replaced by $\overline{\tau}$. If $\overline{\tau} = 0$ then this fact is true. We may assume $0 < \overline{\tau} \le T$. By the definition of $\overline{\tau}$ there exists $u \in C([0,\overline{\tau})\colon Y)$ such that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t) = u(t)$ in Y, uniformly on every compact subinterval of $[0,\overline{\tau})$. Clearly, $\|u(t)\|_Y \le r_0$ and $\varphi(u(t)) \le \alpha_0$ for $t \in [0,\overline{\tau})$. We have by Lemma 3.4,

$$Su(t) = U(t,s)Su(s) + \int_s^t U(t,r)B(u(r))Su(r)\,dr$$

for $0 \le s \le t < \overline{\tau}$. Let $0 \le s < \overline{\tau}$ and $s \le t, \hat{t} < \overline{\tau}$. We find by the equality above

$$\|S(u(t) - u(\hat{t}))\|_Z \le \|U(t,s)Su(s) - U(\hat{t},s)Su(s)\|_Z + ((t-s) + (\hat{t}-s))C,$$

where $C = M(\alpha_0)\exp(\beta_0 T)M_B(\alpha_0)\|S\|_{Y,Z}r_0$. By (ii) of Lemma 3.5 we see that the first term on the right-hand side vanishes as $t, \hat{t} \uparrow \overline{\tau}$. As $t, \hat{t} \uparrow \overline{\tau}$, the last term converges to $2C(\overline{\tau} - s)$, which tends to zero as $s \uparrow \overline{\tau}$. This proves that the limit $\lim_{t\uparrow\overline{\tau}} u(t)$ exists in Y, and so the desired claim is obtained.

We have only to show $\overline{\tau} = T$ by the fact which was proved in the first part. Assume to the contrary that $\overline{\tau} < T$. By what we have just proved, there exists $u \in C([0,\overline{\tau}]\colon Y)$ satisfying the intergal equation

$$Su(t) = U(t,0)Su_0 + \int_0^t U(t,r)B(u(r))Su(r)\,dr \tag{3.13}$$

for $t \in [0,\overline{\tau}]$. By a fixed point argument one finds a $\delta > 0$ so that the integral equation

$$Sv(t) = U(t,\overline{\tau})Su(\overline{\tau}) + \int_{\overline{\tau}}^t U(t,r)B(v(r))Sv(r)\,dr \tag{3.14}$$

has a unique solution $v \in C([\overline{\tau}, \overline{\tau}+\delta]\colon Y)$. (See also [9, Lemmas 3.5 and 3.6].) Substituting (3.13) with $t = \overline{\tau}$ into (3.14) and using (iii) of Lemma 3.5 we have

$$Sv(t) = U(t,0)Su_0 + \int_0^{\overline{\tau}} U(t,r)B(u(r))Su(r)\,dr + \int_{\overline{\tau}}^t U(t,r)B(v(r))Sv(r)\,dr$$

for $t \in [\overline{\tau}, \overline{\tau}+\delta]$. It follows that u can be extended to an element of $C([0,\overline{\tau}+\delta]: Y)$ which satisfies (3.11) with $\tau = \overline{\tau}+\delta$, by defining $u(t) = v(t)$ for $t \in [\overline{\tau}, \overline{\tau}+\delta]$. By the first part of the proof we see that u satisfies (3.10) with $\tau = \overline{\tau}+\delta$, which is a contradiction to the definition of $\overline{\tau}$. ∎

Proof of main theorem: Let $u_0 \in Y_{\alpha_0}$ and $T > 0$. Since $\lim_{\varepsilon\downarrow 0} \tau_\varepsilon(\varphi(u_0)) = \tau(\varphi(u_0)) = \infty$, there is an $\varepsilon_0 \in (0,1]$ so that $T+1 < \tau_\varepsilon(\varphi(u_0))$ for $\varepsilon \in (0,\varepsilon_0]$. By Theorem 2.1, for each $\varepsilon \in (0,\varepsilon_0]$ there exist a sequence $\{t_i^\varepsilon\}_{i=0}^{N_\varepsilon}$ of nonnegative numbers and a sequence $\{u_i^\varepsilon\}_{i=0}^{N_\varepsilon}$ in Y such that they satisfy conditions (i) through (iii) of Theorem 3.1 and

$$\varphi(u_i^\varepsilon) \le m_\varepsilon(t_i^\varepsilon; \varphi(u_0)) \tag{3.15}$$

for $i = 1,2,\dots,N_\varepsilon$. Since $m_\varepsilon(t;\varphi(u_0)) \downarrow m(t;\varphi(u_0))$ uniformly on $[0,T+1]$ as $\varepsilon \downarrow 0$ (by (ii) of Proposition 1.2), we have

$$\alpha_0 = \sup\{m_\varepsilon(t;\varphi(u_0)) \colon \varepsilon \in (0,\varepsilon_0] \text{ and } t \in [0,T+1]\} < \infty.$$

It follows that $\varphi(u_i^\varepsilon) \le \alpha_0$ for $i = 1,2,\dots,N_\varepsilon$ and $\varepsilon \in (0,\varepsilon_0]$. Condition ($\varphi$1) implies that statement (a) of Theorem 3.1 is true. From Theorem 3.1 we deduce that the (QE) has a classical solution u on $[0,T]$ satisfying (3.1). By (3.15) we have $\varphi(u(t)) \le m(t;\varphi(u_0))$ for $t \in [0,T]$. Since $T > 0$ is arbitrary, the desired claim follows from standard arguments together with Theorem 1.1. ∎

4. Applications to hyperbolic equations

In this section we shall apply our abstract theory to Cauchy problems of three nonlinear hyperbolic equations.

Let us first consider the hyperbolic equation

$$u''(t) + \mathcal{A}^2 u(t) + \beta'(|\mathcal{A}^{1/2}u(t)|^2)\mathcal{A}u(t) = 0 \quad \text{for } t \ge 0 \tag{4.1}$$

in a real Hilbert space $\mathcal{H}$ with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $|\cdot|$. Here $\mathcal{A}$ is a positive selfadjoint operator in $\mathcal{H}$, and so there is a $c > 0$ such that $\langle \mathcal{A}u, u\rangle \ge c|u|^2$ for $u \in D(\mathcal{A})$. It is assumed that $\beta \in C^2([0,\infty) \colon \mathbb{R})$ satisfies $\beta(0) = 0$ and $\beta'(r) \ge m_0 > 0$ for $r \ge 0$. This is the abstract version of damped extensible beam equations (see Patcheu [13]).

By $\mathcal{V}$ and $\mathcal{W}$ we denote real Hilbert spaces $D(\mathcal{A})$ and $D(\mathcal{A}^2)$ equipped with the inner products $\langle u,\hat{u}\rangle_{\mathcal{V}} = \langle \mathcal{A}u, \mathcal{A}\hat{u}\rangle$ for $u,\hat{u} \in \mathcal{V}$, and $\langle u,\hat{u}\rangle_{\mathcal{W}} = \langle \mathcal{A}^2u, \mathcal{A}^2\hat{u}\rangle$ for

$u, \hat{u} \in \mathcal{W}$, respectively. We shall prove that for each $(\phi_0, \psi_0) \in \mathcal{W} \times \mathcal{V}$, problem (4.1) has a unique solution u in the class

$$C^2([0,\infty)\colon \mathcal{W}) \cap C^1([0,\infty)\colon \mathcal{V}) \cap C([0,\infty)\colon \mathcal{H})$$

satisfying the initial condition $(u(0), u'(0)) = (\phi_0, \psi_0)$. To this end, let $(\phi_0, \psi_0) \in \mathcal{W} \times \mathcal{V}$ and set $E_0 = |\psi_0|^2 + |\mathcal{A}\phi_0|^2 + \beta(|\mathcal{A}^{1/2}\phi_0|^2)$. We first note that a function u defined on $[0,\infty)$ is a solution of (4.1) satisfying the initial condition $(u(0), u'(0)) = (\phi_0, \psi_0)$ if and only if it is a solution of

$$u''(t) + \mathcal{A}^2 u(t) + \tilde{\beta}'(|\mathcal{A}^{1/2}u(t)|^2)\mathcal{A}u(t) = 0 \quad \text{for } t \geq 0 \tag{4.2}$$

with the initial condition $(u(0), u'(0)) = (\phi_0, \psi_0)$, where $\tilde{\beta}$ is defined by

$$\tilde{\beta}(r) = \int_0^r \beta'(s \wedge (E_0/m_0))\, ds$$

for $r \geq 0$. Indeed, if u is a solution of (4.2) on $[0,\infty)$ then

$$\frac{d}{dt}\left(|u'(t)|^2 + |\mathcal{A}u(t)|^2 + \tilde{\beta}(|\mathcal{A}^{1/2}u(t)|^2)\right) = 0 \qquad \text{for } t \geq 0$$

which follows easily by taking the inner product (4.2) with $2u'(t)$. By the assumption of β we have $|\mathcal{A}^{1/2}\phi_0|^2 \leq E_0/m_0$ and $\tilde{\beta}(r) \geq m_0 r$ for $r \geq 0$; hence

$$m_0|\mathcal{A}^{1/2}u(t)|^2 \leq |\psi_0|^2 + |\mathcal{A}\phi_0|^2 + \tilde{\beta}(|\mathcal{A}^{1/2}\phi_0|^2) = E_0$$

for $t \geq 0$, by which we see that u is a solution of (4.1) on $[0,\infty)$. The converse is proved in the same way.

Now, we shall study the Cauchy problem for (4.2). For this purpose, we convert the differential equation (4.2) into the first order system

$$(d/dt)(u(t), v(t)) = (A(u(t), v(t)))(u(t), v(t)) \quad \text{for } t \geq 0$$

in the Hilbert space Z $(= X) = \mathcal{V} \times \mathcal{H}$, where $\{A(w,z)\colon (w,z) \in Y\}$ is a family of linear operators in Z defined by

$$(A(w,z))(u,v) = (v, -\mathcal{A}^2 u - \tilde{\beta}'(|\mathcal{A}^{1/2}w|^2)\mathcal{A}u)$$

for $(u,v) \in D(A(w,z)) = Y := \mathcal{W} \times \mathcal{V}$. By the positivity of A we have $c|u| \leq |\mathcal{A}u|$ for $u \in D(\mathcal{A})$; hence $|\mathcal{A}^{1/2}u|^2 \leq \langle \mathcal{A}u, u\rangle \leq |\mathcal{A}u|^2/c$ for $u \in D(\mathcal{A})$, and $c^2|u| \leq$

$c|\mathcal{A}u| \le |\mathcal{A}^2u|$ for $u \in D(\mathcal{A}^2)$, by which we see that Y is continuously imbedded in X, and find two inequalities

$$|w|^2 \le \alpha/c^4, \tag{4.3}$$

$$|\mathcal{A}^{1/2}w|^2 \le \alpha/c^3 \tag{4.4}$$

for $(w,z) \in Y_\alpha$. Let us define a functional φ on Y by

$$\varphi(u,v) = |u|_{\mathcal{W}}^2 + |v|_{\mathcal{V}}^2 \ (= \|(u,v)\|_Y^2)$$

for $(u,v) \in Y$. Clearly, φ is continuous on Y and satisfies conditions (φ1) and (φ2). We introduce a family $\{\langle\cdot,\cdot\rangle_{(w,z)} : (w,z) \in Y\}$ of inner products in Z defined by

$$\langle (u,v),(\hat{u},\hat{v})\rangle_{(w,z)} = \tilde{\beta}'(|\mathcal{A}^{1/2}w|^2)\langle \mathcal{A}^{1/2}u, \mathcal{A}^{1/2}\hat{u}\rangle + \langle \mathcal{A}u, \mathcal{A}\hat{u}\rangle + \langle v, \hat{v}\rangle$$

for $(u,v),(\hat{u},\hat{v}) \in Z$. Condition (N1) is easily checked by (4.4). To prove (N2) let $(u,v) \in Z$ with $(u,v) \ne (0,0)$, and $(w,z),(\hat{w},\hat{z}) \in Y_\alpha$. Then we have

$$\begin{aligned}\frac{\|(u,v)\|_{(w,z)}}{\|(u,v)\|_{(\hat{w},\hat{z})}} &= \frac{\|(u,v)\|_{(w,z)}^2 - \|(u,v)\|_{(\hat{w},\hat{z})}^2}{(\|(u,v)\|_{(w,z)} + \|(u,v)\|_{(\hat{w},\hat{z})})\|(u,v)\|_{(\hat{w},\hat{z})}} + 1 \\ &\le \frac{\tilde{\beta}'(|\mathcal{A}^{1/2}w|^2) - \tilde{\beta}'(|\mathcal{A}^{1/2}\hat{w}|^2)}{2m_0} + 1.\end{aligned}$$

By (4.3) and (4.4) we have

$$\begin{aligned}&|\tilde{\beta}'(|\mathcal{A}^{1/2}w|^2) - \tilde{\beta}'(|\mathcal{A}^{1/2}\hat{w}|^2)| \\ &\le L_{\tilde{\beta}'}(|\mathcal{A}^{1/2}w|^2 \vee |\mathcal{A}^{1/2}\hat{w}|^2)|\langle \mathcal{A}(w-\hat{w}), w+\hat{w}\rangle| \\ &\le 2\alpha^{1/2}L_{\tilde{\beta}'}(\alpha/c^3)\|w-\hat{w}\|_V/c^2,\end{aligned} \tag{4.5}$$

where $L_{\tilde{\beta}'}(\tau)$ denotes the Lipschitz constant of $\tilde{\beta}'$ on $[0,\tau]$. It follows that condition (N2) is satisfied.

To prove condition (A1), let $(w,z) \in Y$. A straightforward computation yields $\langle (u,v),(A(w,z))(u,v)\rangle_{(w,z)} = 0$ for $(u,v) \in Y$. To prove the range condition, let $(f,g) \in \mathcal{V} \times \mathcal{H}$ and $\lambda > 0$. A bounded bilinear form $a[w;\cdot,\cdot] : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by

$$a[w;u,\hat{u}] = \langle u,\hat{u}\rangle + \lambda^2\langle \mathcal{A}u, \mathcal{A}\hat{u}\rangle + \lambda^2\tilde{\beta}'(|\mathcal{A}^{1/2}w|^2)\langle \mathcal{A}^{1/2}u, \mathcal{A}^{1/2}\hat{u}\rangle$$

satisfies the estimate $a[w;u,u] \ge \lambda^2|u|_{\mathcal{V}}^2$ which means that $a[w;\cdot,\cdot]$ is coercive on $\mathcal{V} \times \mathcal{V}$. A functional F on $\mathcal{V}$ defined by $F(\hat{u}) = \langle f,\hat{u}\rangle + \lambda\langle g,\hat{u}\rangle$ is linear and bounded. By Lax–Milgram's theorem there exists $u \in \mathcal{V}$ such that $a[w; u, \hat{u}] = F(\hat{u})$ for all $\hat{u} \in \mathcal{V}$; namely

$$\langle u, \hat{u}\rangle + \lambda^2\langle \mathcal{A}u, \mathcal{A}\hat{u}\rangle + \lambda^2\tilde{\beta}'(|\mathcal{A}^{1/2}w|^2)\langle \mathcal{A}^{1/2}u, \mathcal{A}^{1/2}\hat{u}\rangle = \langle f, \hat{u}\rangle + \lambda\langle g, \hat{u}\rangle$$

for all $\hat{u} \in \mathcal{V}$. Put $v = (u - f)/\lambda \in \mathcal{V}$. By the equality above we find

$$\langle \mathcal{A}u, \mathcal{A}\hat{u}\rangle = \langle (g - v)/\lambda - \tilde{\beta}'(|\mathcal{A}^{1/2}w|^2)\mathcal{A}u, \hat{u}\rangle$$

for $\hat{u} \in \mathcal{V}$, which implies

$$\mathcal{A}u \in D(\mathcal{A}^*) = D(\mathcal{A}) \quad \text{and} \quad \mathcal{A}^2 u = (g - v)/\lambda - \tilde{\beta}'(|\mathcal{A}^{1/2}w|^2)\mathcal{A}u.$$

It follows that $R(I - \lambda A(w, z)) = Z$ for all $\lambda > 0$. It has been proved that condition (A1) holds with $\omega(\alpha) = 0$. Condition (A2) is satisfied with $B(w, z) = 0$ by choosing the isomorphism

$$S = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A} \end{pmatrix}$$

of Y onto Z. By (4.5) we obtain the desired inequality (1.5), which implies the fact that $A(\cdot) \in C(Y; B(Y, X))$ and (1.4), since $X = Z$ in this problem.

We shall check condition (G). To this end, let $(u_0, v_0) \in \mathcal{W} \times \mathcal{V}$ and set

$$(u_\lambda, v_\lambda) = (I - \lambda A(u_0, v_0))^{-1}(u_0, v_0). \tag{4.6}$$

By Remark 1.1 there exists $\lambda_0 > 0$ such that $(u_\lambda, v_\lambda) \in \mathcal{W} \times \mathcal{V}$ for $\lambda \in (0, \lambda_0]$ and $(u_\lambda, v_\lambda) \to (u_0, v_0)$ in $\mathcal{W} \times \mathcal{V}$ as $\lambda \downarrow 0$. (4.6) is written as

$$(u_\lambda - u_0)/\lambda = v_\lambda, \tag{4.7}$$

$$(v_\lambda - v_0)/\lambda + \mathcal{A}^2 u_\lambda + \tilde{\beta}'(|\mathcal{A}^{1/2}u_0|^2)\mathcal{A}u_\lambda = 0. \tag{4.8}$$

By (4.7) we have $v_\lambda \in \mathcal{W}$; hence $\mathcal{A}^2 v_\lambda$ makes sense. Taking the inner product of (4.8) with $\mathcal{A}^2 v_\lambda$ we have by (4.7),

$$\langle \mathcal{A}v_\lambda, \mathcal{A}v_\lambda - \mathcal{A}v_0\rangle/\lambda + \langle \mathcal{A}^2 u_\lambda - \mathcal{A}^2 u_0, \mathcal{A}^2 u_\lambda\rangle/\lambda + \tilde{\beta}'(|\mathcal{A}^{1/2}u_0|)\langle \mathcal{A}v_\lambda, \mathcal{A}^2 u_\lambda\rangle = 0.$$

By the inequality

$$(|u|^2 - |v|^2)/2 \le \langle u, u - v\rangle \tag{4.9}$$

for $u, v \in \mathcal{H}$, we have

$$(|\mathcal{A}v_\lambda|^2 - |\mathcal{A}v_0|^2)/\lambda + (|\mathcal{A}^2u_\lambda|^2 - |\mathcal{A}^2u_0|^2)/\lambda \le 2\tilde{\beta}'(|\mathcal{A}^{1/2}u_0|^2)|\mathcal{A}v_\lambda||\mathcal{A}^2u_\lambda|.$$

An application of Young's inequality gives

$$(\varphi(u_\lambda, v_\lambda) - \varphi(u_0, v_0))/\lambda \le a\varphi(u_\lambda, v_\lambda),$$

where $a = \sup\{\tilde{\beta}'(r) : r \ge 0\} < \infty$. This implies that condition (G) is satisfied with a comparison function g defined by $g(r) = ar$ for $r \ge 0$, since $(u_\lambda, v_\lambda) \to (u_0, v_0)$ in $\mathcal{W} \times \mathcal{V}$ as $\lambda \downarrow 0$.

We next consider the quasi-linear wave equation of Kirchhoff type

$$u''(t) + \beta'(|\mathcal{A}^{1/2}u(t)|^2)\mathcal{A}u(t) + \nu u'(t) = 0 \qquad \text{for } t \ge 0 \tag{4.10}$$

in a real Hilbert space $\mathcal{H}$ with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $|\cdot|$. It is assumed that $\mathcal{A}$ is a nonnegative selfadjoint operator in $\mathcal{H}$, $\nu > 0$ and that $\beta \in C^2([0,\infty) : \mathbb{R})$ satisfies $\beta(0) = 0$ and $\beta'(r) \ge m_0 > 0$ for $r \ge 0$. This problem has been intensively investigated. (For example, see Heard [3] and Ono [12].)

We apply the main theorem to prove the existence and uniqueness of a global solution u of (4.10) in the class

$$C([0,\infty) : \mathcal{W}) \cap C^1([0,\infty) : \mathcal{V}) \cap C^2([0,\infty) : \mathcal{H}),$$

where $\mathcal{W}$ and $\mathcal{V}$ are two real Hilbert spaces $D(\mathcal{A})$ and $D(\mathcal{A}^{1/2})$ equipped with inner products $\langle u, \hat{u}\rangle_{\mathcal{W}} = \langle u, \hat{u}\rangle + \langle \mathcal{A}u, \mathcal{A}\hat{u}\rangle$ for $u, \hat{u} \in \mathcal{W}$, and $\langle u, \hat{u}\rangle_{\mathcal{V}} = \langle u, \hat{u}\rangle + \langle \mathcal{A}^{1/2}u, \mathcal{A}^{1/2}\hat{u}\rangle$ for $u, \hat{u} \in \mathcal{V}$, respectively.

Now, we take $Z = X = \mathcal{V} \times \mathcal{H}$ and $Y = \mathcal{W} \times \mathcal{V}$. Clearly, Y is continuously imbedded in X, since $|\mathcal{A}^{1/2}u|^2 = \langle \mathcal{A}u, u\rangle \le |u|_{\mathcal{W}}^2/2$ for $u \in D(\mathcal{A})$. (4.10) is converted into the first order system in Z

$$(d/dt)(u(t), v(t)) = (A(u(t), v(t)))(u(t), v(t)) \qquad \text{for } t \ge 0.$$

Here $\{A(w,z) : (w,z) \in Y\}$ is a family of linear operators in Z defined by

$$(A(w,z))(u,v) = (v, -\beta'(|\mathcal{A}^{1/2}w|^2)\mathcal{A}u - \nu v)$$

for $(u,v) \in D(A(w,z)) := Y$. Let us consider a functional φ on Y by

$$\varphi(u,v) = (|\nu u + v|^2 + |v|^2 + 2\beta(|\mathcal{A}^{1/2}u|^2) + |\mathcal{A}^{1/2}(\nu u + v)|^2 + \beta'(|\mathcal{A}^{1/2}u|^2)|\mathcal{A}u|^2)/2$$

for $(u,v) \in Y$. Clearly, φ is continuous on Y. By the assumption of β we have $m_0 r \le \beta(r) \le M_{\beta'}(r)r$ for $r \ge 0$, where $M_{\beta'}(r) = \sup\{\beta'(s) \colon s \in [0,r]\}$ for $r \ge 0$. It follows that

$$(1 \wedge m_0)\|(u,v)\|^2/2 \le \varphi(u,v) \le (1 \vee M_{\beta'}(|\mathcal{A}^{1/2}u|^2))\|(u,v)\|^2/2$$

for $(u,v) \in Y$. Here $\|(u,v)\|$ is a norm in Y defined by

$$\|(u,v)\| = (|\nu u + v|^2 + |v|^2 + 2|\mathcal{A}^{1/2}u|^2 + |\mathcal{A}^{1/2}(\nu u + v)|^2 + |\mathcal{A}u|^2)^{1/2},$$

which is equivalent to the norm $\|(u,v)\|_Y$; hence there are $C_\nu \ge c_\nu > 0$ such that

$$c_\nu \|(u,v)\|_Y^2 \le \varphi(u,v) \le C_\nu (1 \vee M_{\beta'}(|\mathcal{A}^{1/2}u|^2))\|(u,v)\|_Y^2$$

for $(u,v) \in Y$. This means that φ satisfies conditions (φ1) and (φ2). We introduce a family $\{\|\cdot\|_{(w,z)} \colon (w,z) \in Y\}$ of norms in Z defined by $\|(u,v)\|_{(w,z)} = \langle (u,v),(u,v)\rangle_{(w,z)}^{1/2}$ where

$$\langle (u,v),(\hat{u},\hat{v})\rangle_{(w,z)} = \beta'(|\mathcal{A}^{1/2}w|^2)\langle \mathcal{A}^{1/2}u, \mathcal{A}^{1/2}\hat{u}\rangle + \langle u,\hat{u}\rangle + \langle v,\hat{v}\rangle$$

for $(u,v),(\hat{u},\hat{v}) \in Z$. By an argument similar to that in the first example, it is shown that the family $\{\|\cdot\|_{(w,z)} \colon (w,z) \in Y\}$ satisfies condition (N) and that the family $\{A(w,z) \colon (w,z) \in Y\}$ satisfies condition (A) by taking

$$S = \begin{pmatrix} (I+\mathcal{A})^{1/2} & 0 \\ 0 & (I+\mathcal{A})^{1/2} \end{pmatrix}$$

as an isomorphism of Y onto Z. Here we note that the range condition is checked by considering a bilinear form $a[w;\cdot,\cdot]$ on $\mathcal{V} \times \mathcal{V}$ defined by

$$a[w;u,\hat{u}] = (1+\lambda\nu)\langle u,\hat{u}\rangle + \lambda^2\beta'(|\mathcal{A}^{1/2}w|^2)\langle \mathcal{A}^{1/2}u, \mathcal{A}^{1/2}\hat{u}\rangle$$

and a linear functional F on $\mathcal{V}$ defined by $F(\hat{u}) = (1+\lambda\nu)\langle f,\hat{u}\rangle + \lambda\langle g,\hat{u}\rangle$.

It remains to check condition (G). To do so, let $(u_0,v_0) \in Y$ and define (u_λ,v_λ) by (4.6). By Remark 1.1, there exists $\lambda_0 > 0$ such that $(u_\lambda,v_\lambda) \in Y$ for $\lambda \in (0,\lambda_0]$ and $(u_\lambda,v_\lambda) \to (u_0,v_0)$ in Y as $\lambda \downarrow 0$. (4.6) is written as

$$u_\lambda - u_0 = \lambda v_\lambda, \tag{4.11}$$

$$v_\lambda - v_0 + \lambda\beta'(|\mathcal{A}^{1/2}u_0|^2)\mathcal{A}u_\lambda + \lambda\nu v_\lambda = 0. \tag{4.12}$$

Taking the inner product of (4.12) with v_λ we have, by (4.11),

$$\langle v_\lambda, v_\lambda - v_0\rangle + \beta'(|\mathcal{A}^{1/2}u_0|^2)\langle \mathcal{A}u_\lambda, u_\lambda - u_0\rangle + \lambda\nu|v_\lambda|^2 = 0. \tag{4.13}$$

We take inner products of (4.11) and (4.12) with $\nu^2 u_\lambda$ and νu_λ respectively, and sum up the resultant equalities. This yields

$$\nu^2\langle u_\lambda - u_0, u_\lambda\rangle + \nu\langle v_\lambda - v_0, u_\lambda\rangle + \lambda\nu\beta'(|\mathcal{A}^{1/2}u_0|^2)|\mathcal{A}^{1/2}u_\lambda|^2 = 0.$$

Addition of this and (4.13) gives

$$\langle \nu u_\lambda + v_\lambda, \nu(u_\lambda - u_0) + (v_\lambda - v_0)\rangle + \beta'(|\mathcal{A}^{1/2}u_0|^2)\langle \mathcal{A}u_\lambda, u_\lambda - u_0\rangle \le 0.$$

By this inequality and (4.13) we have

$$(|\varepsilon\nu u_\lambda + v_\lambda|^2 - |\varepsilon\nu u_0 + v_0|^2)/2 + \beta'(|\mathcal{A}^{1/2}u_0|^2)(|\mathcal{A}^{1/2}u_\lambda|^2 - |\mathcal{A}^{1/2}u_0|^2)/2 \le 0$$

for each $\varepsilon = 0, 1$. Note that if $\sigma \in C^1([0,\infty)\colon \mathbb{R})$ then

$$\begin{aligned}&\sigma(|\mathcal{A}^{1/2}u_\lambda|^2) - \sigma(|\mathcal{A}^{1/2}u_0|^2)\\ &\quad = \left(\int_0^1 \sigma'(\theta|\mathcal{A}^{1/2}u_\lambda|^2 + (1-\theta)|\mathcal{A}^{1/2}u_0|^2)\,d\theta\right)\lambda\langle \mathcal{A}(u_\lambda + u_0), v_\lambda\rangle.\end{aligned} \tag{4.14}$$

Making use of (4.14) with $\sigma = \beta$ we have

$$\begin{aligned}&(|\varepsilon\nu u_\lambda + v_\lambda|^2 - |\varepsilon\nu u_0 + v_0|^2)/2\lambda + (\beta(|\mathcal{A}^{1/2}u_\lambda|^2) - \beta(|\mathcal{A}^{1/2}u_0|^2))/2\lambda\\ &\quad + \frac{1}{2}\left(\int_0^1 \{\beta'(|\mathcal{A}^{1/2}u_0|^2) - \beta'(\theta|\mathcal{A}^{1/2}u_\lambda|^2 + (1-\theta)|\mathcal{A}^{1/2}u_0|^2)\}\,d\theta\right)\\ &\qquad \times \langle \mathcal{A}(u_\lambda + u_0), v_\lambda\rangle \le 0\end{aligned} \tag{4.15}$$

for each $\varepsilon = 0, 1$. By (4.11) we have $v_\lambda = (u_\lambda - u_0)/\lambda \in D(\mathcal{A})$; hence $\mathcal{A}v_\lambda$ makes sense. We take inner products of (4.12) with $\mathcal{A}v_\lambda$ and $\nu\mathcal{A}u_\lambda$ respectively and then use (4.11). This yields two equalities

$$\begin{aligned}&\beta'(|\mathcal{A}^{1/2}u_0|^2)\langle \mathcal{A}u_\lambda, \mathcal{A}(u_\lambda - u_0)\rangle\\ &\quad + \langle \mathcal{A}^{1/2}v_\lambda, \mathcal{A}^{1/2}(v_\lambda - v_0) + \nu\mathcal{A}^{1/2}(u_\lambda - u_0)\rangle = 0\end{aligned}$$

and

$$\langle \nu\mathcal{A}^{1/2}u_\lambda, \mathcal{A}^{1/2}(v_\lambda - v_0) + \nu\mathcal{A}^{1/2}(u_\lambda - u_0)\rangle + \lambda\nu\beta'(|\mathcal{A}^{1/2}u_0|^2)|\mathcal{A}u_\lambda|^2 = 0.$$

Adding these equalities we find

$$(|\nu\mathcal{A}^{1/2}u_\lambda + \mathcal{A}^{1/2}v_\lambda|^2 - |\nu\mathcal{A}^{1/2}u_0 + \mathcal{A}^{1/2}v_0|^2)/2\lambda \\ + \beta'(|\mathcal{A}^{1/2}u_0|^2)(|\mathcal{A}u_\lambda|^2 - |\mathcal{A}u_0|^2)/2\lambda + \nu\beta'(|\mathcal{A}^{1/2}u_0|^2)|\mathcal{A}u_\lambda|^2 \le 0.$$

We write the second term on the left-hand side as

$$(\beta'(|\mathcal{A}^{1/2}u_\lambda|^2)|\mathcal{A}u_\lambda|^2 - \beta'(|\mathcal{A}^{1/2}u_0|^2)|\mathcal{A}u_0|^2)/2\lambda \\ - (\beta'(|\mathcal{A}^{1/2}u_\lambda|^2) - \beta'(|\mathcal{A}^{1/2}u_0|^2))|\mathcal{A}u_\lambda|^2/2\lambda$$

and use (4.14) with $\sigma = \beta'$. This yields

$$(4.16) \quad \begin{aligned} &(|\mathcal{A}^{1/2}(\nu u_\lambda + v_\lambda)|^2 - |\mathcal{A}^{1/2}(\nu u_0 + v_0)|^2)/2\lambda \\ &+ (\beta'(|\mathcal{A}^{1/2}u_\lambda|^2)|\mathcal{A}u_\lambda|^2 - \beta'(|\mathcal{A}^{1/2}u_0|^2)|\mathcal{A}u_0|^2)/2\lambda \\ &+ \Big\{\beta'(|\mathcal{A}^{1/2}u_0|^2)\nu - \frac{1}{2}\Big(\int_0^1 \beta''(\theta|\mathcal{A}^{1/2}u_\lambda|^2 + (1-\theta)|\mathcal{A}^{1/2}u_0|^2)\, d\theta\Big) \\ &\qquad \times \langle \mathcal{A}(u_\lambda + u_0), v_\lambda\rangle\Big\}|\mathcal{A}u_\lambda|^2 \le 0. \end{aligned}$$

Adding (4.15) with $\varepsilon = 0, 1$ and (4.16), and taking the lim inf as $\lambda \downarrow 0$ we find

$$\liminf_{\lambda\downarrow 0}\,(\varphi(u_\lambda, v_\lambda) - \varphi(u_0, v_0))/\lambda \\ + (\beta'(|\mathcal{A}^{1/2}u_0|^2)\nu - \beta''(|\mathcal{A}^{1/2}u_0|^2)\langle \mathcal{A}u_0, v_0\rangle)|\mathcal{A}u_0|^2 \le 0.$$

This means that condition (G) is satisfied with a function g of the form

$$g(r) = ((\rho(r) - m_0\nu)r) \vee 0$$

where ρ is a nonnegative continuous function with $\rho(0) = 0$. It should be noted that the function g defined above is a comparison function, by (ii) of Example 1.1. We therefore conclude that there is an $r_0 > 0$ such that for each $(\phi_0, \psi_0) \in D(\mathcal{A}) \times D(\mathcal{A}^{1/2})$ with $\|(\phi_0, \psi_0)\|_{\mathcal{W}\times\mathcal{V}} \le r_0$, problem (4.10) has a unique solution u in the class

$$C([0,\infty)\colon \mathcal{W}) \cap C^1([0,\infty)\colon \mathcal{V}) \cap C^2([0,\infty)\colon \mathcal{H})$$

satisfying the initial condition $(u(0), u'(0)) = (\phi_0, \psi_0)$.

Finally we give an application of our abstract theory to the Cauchy problem for the quasi-linear wave equation

$$(4.17) \quad \begin{cases} \partial_t u = \partial_x v, \\ \partial_t v = \partial_x \sigma'(u) - \nu v. \end{cases}$$

Here $\nu > 0$ and it is assumed that $\sigma \in C^5(\mathbb{R})$ satisfies $\sigma(0) = \sigma'(0) = 0$ and $\sigma''(r) \geq c_0 > 0$ for $r \in \mathbb{R}$.

To prove the global existence of solutions, we apply the main theorem with three Banach spaces $Z = L^2(\mathbb{R}) \times L^2(\mathbb{R})$, $X = H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and $Y = H^2(\mathbb{R}) \times H^2(\mathbb{R})$. (4.17) is reduced to the abstract evolution equation

$$(d/dt)(u(t), v(t)) = (A(u(t), v(t)))(u(t), v(t)) \quad \text{for } t \geq 0$$

in the real Banach space Z, where $\{A(w, z)\colon (w, z) \in Y\}$ is a family of linear operators in Z defined by

$$(A(w, z))(u, v) = (\partial_x v, \sigma''(w)\partial_x u - \nu v)$$

for $(u, v) \in D(A(w, z)) = H^1(\mathbb{R}) \times H^1(\mathbb{R})$. We use a nonnegative continuous functional φ on Y defined by

$$\begin{aligned}\varphi(u, v) = &\frac{1}{2}\int_{-\infty}^{\infty} (v^2 + |\nu u + \partial_x v|^2 + |\nu\partial_x u + \partial_x^2 v|^2)\, dx \\ &+ \frac{1}{2}\int_{-\infty}^{\infty} \sigma''(u)(|\partial_x u|^2 + |\partial_x^2 u|^2)\, dx + \int_{-\infty}^{\infty} \sigma(u)\, dx.\end{aligned}$$

It is easily seen that

$$c_\nu \|(u, v)\|^2_{H^2 \times H^2} \leq \varphi(u, v) \leq C_\nu(1 \vee M_2(\|u\|_{H^1}))\|(u, v)\|^2_{H^2 \times H^2}$$

for $(u, v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$, where $M_k(r) = \sup\{|\sigma^{(k)}(s)|\colon |s| \leq r\}$ for $r \geq 0$. This means that conditions (φ1) and (φ2) are satisfied. Let us consider a family $\{\langle \cdot, \cdot \rangle_{(w,z)}\colon (w, z) \in Y\}$ of inner products in Z defined by

$$\langle (u, v), (\hat{u}, \hat{v}) \rangle_{(w,z)} = \int_{-\infty}^{\infty} (\sigma''(w(x))u(x)\hat{u}(x) + v(x)\hat{v}(x))\, dx$$

for $(u, v), (\hat{u}, \hat{v}) \in Z$. We recall the following fact needed for later arguments:

$$\text{(4.18)} \qquad H^1(\mathbb{R}) \subset L^\infty(\mathbb{R}) \quad \text{and} \quad \|u\|_\infty \leq \|u\|_{H^1} \text{ for } u \in H^1(\mathbb{R}).$$

If $(w, z) \in Y_\alpha$ then we have by (4.18), $\|\sigma''(w(\cdot))\|_{L^\infty} \leq M_2(\sqrt{\alpha/c_\nu})$ and

$$\text{(4.19)} \qquad \begin{aligned}\|\sigma''(w(\cdot)) - \sigma''(\hat{w}(\cdot))\|_{L^\infty} &\leq M_3(\|w\|_{L^\infty} \vee \|\hat{w}\|_{L^\infty})\|w - \hat{w}\|_{L^\infty} \\ &\leq M_3(\sqrt{\alpha/c_\nu})\|(w, z) - (\hat{w}, \hat{z})\|_X.\end{aligned}$$

These estimates imply condition (N) in a way similar to the derivation in the first example. If $(w,z) \in Y_\alpha$ then $\sigma''(w(\cdot)) \in W^{1,\infty}(\mathbb{R})$ and $\|\partial_x\sigma''(w(\cdot))\|_{L^\infty} \le M_3(\sqrt{\alpha/c_\nu})\sqrt{\alpha/c_\nu}$. By a routine computation we see that for each $(w,z) \in Y_\alpha$, $A(w,z) - \omega(\alpha)I$ is dissipative in $Z_{(w,z)}$ where $\omega(\alpha) = M_3(\sqrt{\alpha/c_\nu})\sqrt{\alpha/c_\nu}/2\sqrt{c_0}$. Similarly to the preceding examples, the range condition is checked by using Lax–Milgram's theorem. It follows that for each $(w,z) \in Y_\alpha$,

$$A(w,z) \in G(Z_{(w,z)}, 1, \omega(\alpha)),$$

which proves condition (A1). (See also [8, Proposition 5.7].) To prove condition (A2), we consider the operator S defined by

$$S(u,v) = (u - \partial_x^2 u, v - \partial_x^2 v)$$

for $(u,v) \in Y$, which is an isomorphism of Y onto Z with the inverse $S^{-1}(u,v) = (\mathcal{R}u, \mathcal{R}v)/2$ for $(u,v) \in Z$, where $\mathcal{R}$ is defined by

$$(\mathcal{R}u)(x) = \int_{-\infty}^{\infty} e^{-|x-y|} u(y)\, dy$$

for $u \in L^2(\mathbb{R})$. We note here that for $u \in L^2(\mathbb{R})$,

$$\mathcal{R}u \in H^2(\mathbb{R}) \quad \text{and} \quad \|\partial_x \mathcal{R}u\|_{H^1} \le C\|u\|_{L^2} \tag{4.20}$$

for some $C > 0$. The relation $SA(w,z)S^{-1} = A(w,z) + B(w,z)$ holds with

$$(B(w,z))(u,v) = (0, -\partial_x^2\sigma''(w(x)) \cdot \partial_x(\mathcal{R}u)(x)/2 - \partial_x\sigma''(w(x)) \cdot \partial_x^2(\mathcal{R}u)(x))$$

for $(u,v) \in Z$ and $(w,z) \in Y$. The desired claim that $B(w,z) \in B(Z)$ is proved by (4.20) and the fact that $\partial_x\sigma''(w(\cdot)) = \sigma^{(3)}(w(\cdot))\partial_x w(\cdot) \in L^\infty(\mathbb{R})$ and $\partial_x^2\sigma''(w(\cdot)) = \sigma^{(4)}(w(\cdot))(\partial_x w(\cdot))^2 + \sigma^{(3)}(w(\cdot))\partial_x^2(w(\cdot)) \in L^2(\mathbb{R})$, if $w \in H^2(\mathbb{R})$. To prove (1.3), we estimate the L^∞ norm of the terms involving $\sigma^{(k)}$ and use (4.18). This yields

$$\begin{aligned} &\|\partial_x^2\sigma''(w(\cdot)) - \partial_x^2\sigma''(\hat{w}(\cdot))\|_{L^2} \vee \|\partial_x\sigma''(w(\cdot)) - \partial_x\sigma''(\hat{w}(\cdot))\|_{L^\infty} \\ &\qquad \le C(\|w\|_{H^2} \vee \|\hat{w}\|_{H^2})\|w - \hat{w}\|_{H^2} \end{aligned} \tag{4.21}$$

for $w, \hat{w} \in H^2(\mathbb{R})$. The desired inequality (1.3) is obtained by (4.20) and (4.21). Since $\|\partial_x\sigma''(w(\cdot))\|_{L^\infty} \le M_3(\|w\|_{H^1})\|w\|_{H^2}$, we find $\|A(w,z)\|_{Y,X} \le C(\|w\|_{H^2})$ for $(w,z) \in Y$, which implies (1.4). The fact that $A \in C(Y; B(Y,X))$ follows immediately from the inequality obtained by (4.19) and (4.21) that

$$\|A(w,z) - A(\hat{w},\hat{z})\|_{Y,X} \le C(\|w\|_{H^2} \vee \|\hat{w}\|_{H^2})\|w - \hat{w}\|_{H^2}$$

for $(w,z),(\hat{w},\hat{z}) \in Y$. Since

$$(\sigma''(w(x)) - \sigma''(\hat{w}(x)))\partial_x u(x) = \left(\int_0^1 \sigma^{(3)}(\theta w(x) + (1-\theta)\hat{w}(x))\,d\theta\right)(w(x) - \hat{w}(x))\partial_x u(x),$$

we have, by (4.18),

$$\|(\sigma''(w) - \sigma''(\hat{w}))\partial_x u\|_{L^2} \le M_3(\|w\|_{H^1} \vee \|\hat{w}\|_{H^1})\|\partial_x u\|_{H^1}\|w - \hat{w}\|_{L^2}$$

for $u \in H^2(\mathbb{R})$ and $w, \hat{w} \in H^1(\mathbb{R})$, from which (1.5) follows readily. It is shown [8, Proposition 5.8] that condition (G) is satisfied with a comparison function g of the form (1.12). All assumptions of main theorem are satisfied, and consequently there exists $r_0 > 0$ such that for each $(u_0, v_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ with $\|(u_0, v_0)\|_{H^2\times H^2} \le r_0$, problem (4.17) has a unique solution (u, v) in the class

$$C([0,\infty)\colon H^2(\mathbb{R}) \times H^2(\mathbb{R})) \cap C^1([0,\infty)\colon H^1(\mathbb{R}) \times H^1(\mathbb{R}))$$

satisfying the initial condition $(u(0,x), v(0,x)) = (u_0(x), v_0(x))$ for $x \in \mathbb{R}$.

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